# STRONGLY SUMMABLE ULTRAFILTERS: SOME PROPERTIES AND GENERALIZATIONS

DAVID J. FERNÁNDEZ BRETÓN

A DISSERTATION SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN DEPARTMENT OF MATHEMATICS AND STATISTICS
YORK UNIVERSITY
TORONTO, ONTARIO
DECEMBER 2014

# Abstract

This dissertation focuses on strongly summable ultrafilters, which are ultrafilters that are related to Hindman's theorem in much the same way that Ramsey ultrafilters are related to Ramsey's theorem. Recall that Hindman's theorem states that whenever we partition the set of natural numbers into two (or any finite number of) cells, one of the cells must entirely contain a set of the form $\mathrm{FS}(X)$ for some infinite $X \subseteq \mathbb{N}$ (here $\mathrm{FS}(X)$ is the collection of all finite sums of the form $\sum_{x \in a} x$ where $a \subseteq X$ is finite and nonempty). A nonprincipal ultrafilter on $\mathbb{N}$ is said to be strongly summable if it has a base of sets of the form $\mathrm{FS}(X)$, this is, if $(\forall A \in p)(\exists X \in [\mathbb{N}]^{\aleph_0})(\mathrm{FS}(X) \subseteq A \text{ and } \mathrm{FS}(X) \in p)$. These ultrafilters were first introduced by Hindman, and subsequently studied by people such as Blass, Eisworth, Hindman, Krautzberger, Matet, Protasov and others. Now, from the viewpoint of the definitions, there is nothing special about $\mathbb{N}$, and analogous definitions for $\mathrm{FS}(X)$ and strongly summable ultrafilter can be considered for any semigroup (in the non-abelian case, one must first fix an ordering for $X$ on order-type $\omega$). It is

not immediate, however, that the results that hold for strongly summable ultrafilters on $\mathbb{N}$ are still satisfied in general. Some of the main results of this dissertation are generalizations of these properties for all abelian groups and some non-abelian cases as well. Notably among these, a strongly summable ultrafilter $p$ on an abelian group $G$ has the so-called trivial sums property: whenever $q, r$ are ultrafilters on $G$ such that $q + r = p$, it must be the case that for some $g \in G$, $q = p + g$ and $r = -g + p$ (this is all in the context of the right-topological semigroup $\beta G$ of all ultrafilters on $G$). The other significant result from this dissertation is a consistency result. It has long been known that the existence of strongly summable ultrafilters (on any abelian group) is not provable from the $\mathsf{ZFC}$ axioms, for it implies the existence of P-points. It is also known, however, that (at least on $\mathbb{N}$) the existence of strongly summable ultrafilters follows from the equality of cardinal invariants $\text{cov}(\mathcal{M}) = \mathfrak{c}$, which is equivalent to Martin's axiom restricted to countable forcing notions. We prove here that there exist models of $\mathsf{ZFC}$ that satisfy $\text{cov}(\mathcal{M}) < \mathfrak{c}$ yet there exist strongly summable ultrafilters on all abelian groups. This can be done using iterations, both with finite or with countable support, of $\sigma$-centred forcing notions which resemble Mathias's or Laver's forcing.

To Rocio, Natán, Tizne and Melissa

who stuck with me throughout this adventure.

# Acknowledgements

I would like to start by emphasizing that the following list will always be, as is any acknowledgements section by its very nature, quite incomplete. Moreover, the ordering chosen here is in no way intended to reflect any kind of "priority" or "ranking", as I really am grateful to all mentioned here in a non-quantifiable way.

Let me mention first and foremost my wife Rocio. I simply cannot find the words (in any language I can think of) to thank her for her complete, unrestricted and unconditional support to take on this insane adventure of moving to a strange land (where people speak a very strange language) for a few years (over four of them) just so I could learn a bunch of math. Having her by my side during all these years has made the happy moments even more intense, and the tough moments quite a bit softer. And I am equally grateful to her and to "life" (or to "the Universe", if you will) for having given me the opportunity to embark in an "adventure within the adventure", namely that of learning to be a father while in the middle of my PhD. studies. Obviously I'm not done with learning that just yet, but Natán knows

I do my best every single day.

In a well-defined sense, I think I would not be here if it were not for the help of my former (master's degree) supervisor Fernando Hernández-Hernández. His advice in many matters, and even recommendation letters, during the PhD. application process were very valuable. And he really hit on target when suggesting me who I should choose as my PhD. supervisor, as it turned out to be a great choice of crucial importance for my being able to properly finish my studies, and we were a great fit for each other. While we are at it, I will of course immensely thank my supervisor Juris Steprāns as well, for all of the time that he spent on me (and even some financial support via his NSERC grant) even though in many moments it was not clear at all whether we were actually heading somewhere. His help was of critical importance when trying to find adequate mathematical problems to work on, and once working on them, his suggestions, ideas and encouragement were of utmost usefulness. I certainly hope that we will be able to work together on some mathematical problems again in the future, and am looking forward to that. And I certainly cannot omit the other members of my supervisory committee: Paul Szeptycki and Nantel Bergeron, whom I thank not only for taking the time for reading this dissertation, but also for their help (mathematical or otherwise) throughout every stage of my Doctoral studies. I am especially thankful to my external Defence Committee member Neil Hindman, who was willing to fly to

Toronto even though in the end Destiny (or is it "The Universe"?) prevented him from doing so (but he still Skyped his way into my Defence). His careful reading of this thesis undoubtedly helped greatly improve its quality, although I of course assume full responsibility for any remaining mistakes, inaccuracies or typos (needless to say, it was also a great honour to have "the" Hindman –the one who proved Hindman's Theorem– to be a member of my Committee). And I am also grateful to the two other professors who were members of my Defence Committee and hence also invested time and effort in reading this dissertation and being present while I defended: Ilijas Farah (to whom I am also grateful for his willingness to talk about Mathematics with me throughout my entire PhD.) and George Tourlakis.

I am just as grateful as well to my fellow graduate students at the Department of Mathematics and Statistics, including but not limited to (and here I order them alphabetically to avoid any commitment to other kind of ordering) Alessandro, Branislav, Dan, Francisco, Jiawei, Luigi, Mark, Martín, Martino, Natasha, Oliver, Saeed, Serdar, Victor, Yegor. Some of them were instrumental in my being able to adapt to a new country in the early stages of my Ph. D. studies; while others have helped provide an intellectually stimulating environment which is the necessary condition for ideas to thrive and hatch (and in this respect I should also thank Ari, Chris, Dana, Dani and Jim from U. of T., as well as the several postdocs, who interacted with me through the Toronto Set Theory Seminars –both the Student

Seminar and the “Adult” Seminar). Some of them lie in the intersection of the two aforementioned categories, while all are appreciated dearly.

Finally, it would be impossible for me not to thank the Consejo Nacional de Ciencia y Tecnología (Conacyt) in Mexico, for their financial support via scholarship number 213921/309058.

# Table of Contents

# Preface

*Uno de los hábitos de la mente es la invención de imaginaciones horribles.*

*Ha inventado el Infierno, ha inventado la predestinación al Infierno, ha imaginado las ideas platónicas, la quimera, la esfinge, los anormales números transfinitos (donde la parte no es menos copiosa que el todo), las máscaras, los espejos, las óperas, la teratológica Trinidad: el Padre, el Hijo y el Espectro insoluble, articulados en un solo organismo...*

Jorge Luis Borges, *La biblioteca total.*

Had the above quotation been mine, I would have added "y los ultrafiltros, en especial los fuertemente sumables" among the list of ubiquitous "horrible imaginations". When I started my Ph. D. studies, slightly over four years ago, I used to conceive of a PhD dissertation in Mathematics as mainly an aggregate of theorems (or more precisely, of proofs of theorems), a considerable amount of which are supposed to be original and due to the dissertator. Today, of course, I know much better. I can see how a Dissertation is, above all, the story of a struggle, the struggle of a mathematics-loving person to determine the scope of her theorem-proving skill. It is the end result of a give-and-take back-and-forth process, whereby the dissertator constantly oscillates between trivial problems and too complicated ones, until she finally finds those that are right for her, in the sense that, while being within reach, they do demand a considerable amount of effort and perseverance (stubbornness) on her part to be able to solve them. The PhD journey is thus, in a sense, a particular case of fullfillment of the old Delphian motto: "Know thyself". Throughout this life-changing four-year-long adventure, I was very fortunate to be able to witness my own transition from a person who *learns* mathematics to a person who *does* mathematics, to the extent that I now very much look forward to the new mathematical problems and challenges that await me after the PhD. The old doubts and self-questioning (will I really be able to...?) have receded and now it feels as though the world of Mathematics (more specifically, the Paradise that Cantor constructed for us) has its doors wide open for me to dive in it, rejoicing in the trial-and-error (and very, very seldom, also trial-and-success) process which is the necessary (although by no means sufficient!) condition for mathematical problem-solving. Doing research (in general, but in Mathematics in particular) involves being lost an overwhelming majority of the time spent in such endeavor. For me (as I think for most people), the main offshot of the time spent during my PhD was not so much that I now spend any less time being completely lost, but that

I now feel much more comfortable being lost. Before that, being lost was almost unbearable and at some points I did feel that, in spite of my passionate fondness of Set Theory, the enterprise that I had set myself on was similar to trying to climb an unclimbable mountain. If Jörg Brendle once found himself "strolling through paradise" [7], during the middle stages of my PhD studies I certainly found myself *struggling* through paradise. This documentary evidence of such a struggle goes as follows: in Chapter 1 I state the main definitions and the context (this is, the elementary results and definitions which it is assumed the reader should know) for this work. Chapter 2 is mainly a friendly exposition of some results from my joint paper [13] with Martino Lupini, although I added two more sections at the beginning with some results that highlight the importance of idempotent and strongly summable ultrafilters. Chapter 3 contains the answer to two questions of Hindman, Steprāns and Strauss from [20], as well as the necessary theory and preliminary results to achieve such an answer. Most of the material from this chapter also appears in [11, 12]. Finally, in Chapter 4 I extend some results of Blass and others which will prompt a study of certain forcing extensions and whether there are strongly summable ultrafilters in these extensions. It could also be said, in a sense, that this is the chapter which deals with constructions: we construct strongly summable ultrafilters by using several different forcing techniques.

# 1 Preliminary Remarks

This chapter introduces the definitions for the main concepts that will occupy us throughout this dissertation, and contains a few theorems or lemmas that will be needed in different parts of this work. Most of the results from this chapter are not original, and in several cases we do not write a complete proof (or sometimes even any proof at all) of the claims that we make.

## 1.1 Notation and Terminology

This dissertation is full of standard set-theoretic notation, in most cases similar to what can be found in standard Set Theory textbooks such as [26]. We reserve the lowercase roman letters $p, q, r, u, v$ for ultrafilters, and the uppercase roman letters $A, B, C, D, W, X, Y, Z$, with or without subscripts, will always denote subsets of the semigroup at hand. Lowercase letters $w, x, y, z$ will typically denote elements of the semigroup that is being dealt with, and the "vector" notation will be used for sequences of elements of the semigroup, e.g. $\vec{x} = \langle x_n \big| n < \omega \rangle$. When the sequences are finite, we use the symbol $\frown$ to denote their concatenation, as in $\vec{x} \frown \vec{y}$. If $G$ is a group and $x \in G$, the symbol $o(x)$ will denote the *order* of $x$, i.e. the least natural number $n$ such that $x^n = e$. We make liberal use of the von Neumann ordinals, usually denoted by Greek letters $\alpha, \beta, \gamma, \zeta, \eta, \xi$; thus for two ordinals $\alpha, \beta$, the expressions $\alpha < \beta$ and $\alpha \in \beta$ are interchangeable. Cardinals are nothing but initial ordinals (this is, ordinals that are not in bijection with any of their predecessors) and will typically be denoted by the greek letters $\kappa, \mu, \lambda$. As a particular case of an ordinal (and at the same time, a cardinal), a natural number $n$ is conceived as the set $\{0, \dots, n-1\}$ of its predecessors, with 0 being equal to the empty set $\varnothing$; and $\omega$ denotes the set of finite ordinals, i.e. the set $\mathbb{N} \cup \{0\}$. The lowercase roman letters $i, j, k, l, m, n$, with or without subscript, will be reserved to denote elements of $\omega$. The letters $M$ and $N$, with or without subscripts, will in general be reserved for denoting subsets of $\omega$ (finite or infinite), although occasionally they might denote natural numbers as well (typically, in these situations we will have e.g. that $N$ denotes a natural number that is "much larger" than $n$). The lowercase roman letters $a, b, c, d$, with or without subscript, will stand for elements of $[\omega]^{<\omega}$, i.e. for finite subsets of $\omega$. Given any set $X$, $[X]^n$ will denote the set of subsets of $X$ with $n$ elements, $[X]^{<\omega} = \bigcup\limits_{n<\omega} [X]^n$ will denote the set of finite subsets of $X$, and $[X]^\omega$ denotes the set of infinite subsets of $X$. The set of finite sequences of elements from $X$ is denoted by $X^{<\omega}$ and the set of infinite sequences, by ${}^{\omega}X$.

The cardinal invariant $\mathrm{cov}(\mathcal{M})$ (read "covering of meagre") is the least cardinal for which Martin's Axiom fails at a countable partial order. This is, $\mathrm{cov}(\mathcal{M})$ is the least $\kappa$ such that one can find $\kappa$-many dense subsets of some countable partial order with no filter meeting them all (this notation is explained by the fact that this cardinal is also the least possible number of meagre sets needed to cover all of the real line). Thus the equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ means that Martin's Axiom holds for countable partial orders, whilst the failure of this principle is expressed by the inequality $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$. The invariant $\mathfrak{p}$, on the other hand, is the least cardinality of a centred family without a pseudointersection, and coincides with the least cardinal for which Martin's Axiom fails at a $\sigma$-centred partial (and, as was recently discovered [28, 29], is also equal to the tower number $\mathfrak{t}$). Thus the principle $\mathfrak{p} = \mathfrak{c}$ is nothing but Martin's Axiom for $\sigma$-centred partial orders.

When dealing with abelian groups, one that will be of utmost importance throughout this work is is the so-called circle group, or 1-dimensional torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The reason for its importance is that (as mentioned in [19, p. 123, Section 1], and thoroughly discussed at the beginning of [12, Section 3]) every abelian group can be embedded in a direct sum of circle groups $\bigoplus_{\alpha<\kappa} \mathbb{T}$ (for some infinite cardinal $\kappa$), hence these direct sums will be used ubiquitously in this work. When talking about the group $\mathbb{T}$, we will freely identify real numbers with their corresponding cosets modulo $\mathbb{Z}$, and conversely we will identify elements of $\mathbb{T}$ –which are cosets modulo $\mathbb{Z}$– with any of the elements of $\mathbb{R}$ representing them. It is therefore possible that we might write something like $t = 0$ and really mean that $t \in \mathbb{Z}$, since in the context of working with $\mathbb{T}$, we talk about the real number $t$ when we actually mean the coset of $t$ modulo $\mathbb{Z}$. This should not cause confusion as the context will always clearly indicate whether we are viewing a real number $t$ as a real number or as an element of $\mathbb{T}$. If there is the need to specify a single representative for an element of $\mathbb{T}$, we will pick the unique representative $t$ satisfying $-\frac{1}{2} < t \leq \frac{1}{2}$. In fact, a lot of the time we will be working with direct sums of several copies of $\mathbb{T}$, i.e. with the group $\bigoplus_{\alpha<\kappa} \mathbb{T}$ for some (infinite) cardinal $\kappa$. In this context, given an ordinal $\beta < \kappa$ we will denote the $\beta$-th projection map by $\pi_\beta$.

When doing forcing, we will denote the ground model by $V$, and the forcing relation by $\Vdash$. We force "downwards", this is, for conditions $p, q$ the expression $p \leq q$ means that $p$ extends $q$. We try not to reserve a letter for the $V$-generic filter added by a certain forcing notion, since we will always specify how we are going to denote the relevant generic object.

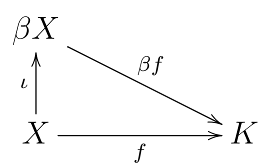
## 1.2 Ultrafilters and the Čech-Stone Compactification

This section contains a very basic and standard introduction to ultrafilters and the Čech-Stone compactification. This can be found with more detail in [21], or in any standard introductory course (and even some introductory textbooks) in General Topology.

Given a nonempty set $X$, a subset $\mathscr{F} \subseteq \mathfrak{P}(X)$ is called a **filter** if it is nonempty (equivalently $X \in \mathscr{F}$, given the next requirement), closed under finite intersections and supersets, and not all of $\mathfrak{P}(X)$ (equivalently $\varnothing \notin \mathscr{F}$). If $\mathscr{F}$ is a filter and it is $\subseteq$-maximal among filters (equivalently, whenever $X = X_0 \cup X_1$ there is $i \in 2$ with $X_i \in \mathscr{F}$), we will say that $\mathscr{F}$ is an **ultrafilter**.

Given a filter $\mathscr{F}$, we say that a subset $\mathscr{B} \subseteq \mathscr{F}$ is a **base** for $\mathscr{F}$ if the latter coincides with the upwards closure of $\mathscr{B}$, in symbols: $\mathscr{F} = \mathscr{B}^+ = \{A \subseteq X \big| (\exists B \in \mathscr{B})(B \subseteq A)\}$. We say that a subset $\mathscr{S} \subseteq \mathscr{F}$ is a **subbase** for $\mathscr{F}$ if the upwards closure of its closure under finite intersections coincides with $\mathscr{F}$, this is, if $\mathscr{F} = \{\bigcap_{B \in \mathcal{F}} B \big| \mathcal{F} \in [\mathscr{S}]^{<\omega}\}^+ = \{A \subseteq X \big| (\exists \mathcal{F} \in [\mathscr{S}]^{<\omega})(\bigcap_{B \in \mathcal{F}} B \subseteq A)\}$. It is easy to prove that every base for a given filter is always a subbase for it too.

We will now say a few words about the Čech-Stone compactification of a topological space. Throughout this dissertation, all hypothesized spaces will be assumed to be Hausdorff. If $X$ is a topological space (when talking about topological spaces, we will usually symbolize their underlying sets only and leave their topologies implicit), a **Čech-Stone compactification** for $X$ is a compact topological space $\beta X$ together with an embedding $\iota : X \longrightarrow \beta X$ (this is, $\iota$ is an injective continuous mapping and it is a homeomorphism between $X$ and $\iota[X]$) such that $\iota[X]$ is dense in $\beta X$ and that satisfies the following universal property: whenever $K$ is a compact and $f : X \longrightarrow K$ is a continuous function, there exists a unique continuous function $\beta f : \beta X \longrightarrow K$ that renders the following diagram commutative:




In this case, we typically think of $X$ as being a dense subspace of $\beta X$, and of $\iota$ as being the inclusion mapping. The above universal property implies that, if $X$ has any Čech-Stone compactification at all, then that compactification is unique up to a homeomorphism fixing $X$. It is possible to prove that a topological space has a Čech-Stone compactification if and only if it is $T_{3\frac{1}{2}}$. Hence it can be verified that the symbol $\beta$ actually denotes a covariant functor from the category of $T_{3\frac{1}{2}}$ spaces (also known as completely regular spaces, or Tychonov spaces, or uniformizable spaces) to the category of compact Hausdorff spaces (where we consider all continuous mappings to be the arrows for both categories).

When $X$ is just a "bare" set, without any topological structure, we will automatically equip it with the discrete topology. In this case, it is possible to realize the Čech-Stone compactification $\beta X$ of $X$ as the set of all ultrafilters on $X$ equipped with the topology that has as a basis the collection of all sets of the form

$\bar{A} = \{p \in \beta X \big| A \in p\}$, for $A \subseteq X$; and the inclusion mapping $\iota$ that sends every $x \in X$ to its corresponding principal ultrafilter, i.e. $\iota(x) = \{A \subseteq X \big| x \in A\}$. Under these circumstances, what we originally denoted by $\bar{A}$ is really the closure in $\beta X$ of the set $\iota[A]$ (which we identify with $A$ itself).

If $\mathscr{F}$ is a filter, it is not hard to see that $\widehat{\mathscr{F}} = \{p \in \beta X \big| \mathscr{F} \subseteq p\}$ is a closed subset of $\beta X$. Conversely, if $F \subseteq \beta X$ is a closed set then $\bigcap F$ is a filter, and it is possible to check that $\widehat{\bigcap F} = F$ and $\bigcap \widehat{\mathscr{F}} = \mathscr{F}$. So it is possible to identify filters on $X$ with closed subsets of $\beta X$. A good example is the so-called **Čech-Stone remainder** $X^* = \beta X \setminus X$ which equals $\widehat{\mathscr{F}}$ where $\mathscr{F}$ is the Fréchet filter consisting of all cofinite subsets of $X$.

There is also a neat characterization of the continuous extension of functions. Whenever we have a function $f : X \longrightarrow Y$ between two sets, if $p$ is an ultrafilter in $X$ then the family $f(p) = \{A \subseteq Y \big| f^{-1}[A] \in p\}$ is an ultrafilter on $Y$, for which the family $\{f[A] \big| A \in p\}$ is a base. Now if $K$ is a compact Hausdorff space, and $f : X \longrightarrow K$ is any mapping (which is automatically continuous as $X$ is discrete), then the continuous extension $\beta f : \beta X \longrightarrow K$ given by the universal property is as follows: $(\beta f)(p)$ is the (unique, as $K$ is Hausdorff) limit of the ultrafilter $f(p)$ in $K$ (which exists since $K$ is compact).

In particular, we can get a nice description of $\beta$ as a functor, when its domain is restricted to the category of discrete spaces (which is really just the category of sets). For any mapping $f : X \longrightarrow Y$ between two sets, one can think of it as a mapping $f : X \longrightarrow \beta Y$, in which case $(\beta f)(p)$ is just $f(p)$ (since the limit in $\beta Y$ of any ultrafilter on $Y$ is itself). The ultrafilter $f(p)$ is called the *Rudin-Keisler image of $p$ under $f$.*

We will now turn our attention to semigroups. If $(S, *)$ is a semigroup, to its algebraic structure we will add a topological one by, again, equipping $S$ with the discrete topology. This turns $S$ into a topological semigroup, meaning that the semigroup operation $* : S \times S \longrightarrow S$ is continuous. We will now extend the semigroup operation to all of $\beta S$ as follows. First we consider, for any $x \in S$, the left translation $\lambda_x : S \longrightarrow S \subseteq \beta S$ given by $\lambda_x(y) = x * y$. Then the unique extension $\beta\lambda_x : \beta S \longrightarrow \beta S$ allows us to define the product of an element $x \in S$ by an ultrafilter $p \in \beta S$ as $x * p = (\beta\lambda_x)(p)$. That allows us to define, for every $p \in \beta S$, the right-translation function $\rho_p : S \longrightarrow \beta S$ by $\rho_p(x) = x * p$, whereby the continuous extension $\beta\rho_p : \beta S \longrightarrow \beta S$ gives us a way of multiplying any two ultrafilters $q, p \in \beta S$ by defining $q * p = (\beta\rho_p)(q)$. If we calculate what this means in purely combinatorial terms, we get the following formula for the product of two ultrafilters:

$$p * q = \{A \subseteq S \big| \{x \in S \big| \{y \in S \big| x * y \in A\} \in q\} \in p\}.$$

Usually we will, for short, denote $x^{-1} * A = \{y \in S \big| x * y \in A\}$, so that $p * q = \{A \subseteq$

$S|\{x \in S|x^{-1} * A \in q\} \in p\}$. It is useful to note that this product is just a particular case of the so-called *Arens product* in the Banach algebra $\ell_\infty(S)^*$. Without diving into the deep details of this construction, we can just say that, if we think of an ultrafilter as a two-valued measure on $S$ (given by assigning any $A \subseteq S$ measure 1 if and only if $A$ belongs to the given ultrafilter, and measure zero otherwise), then $p * q$ is the new measure which is given by assigning to any $A \subseteq S$ the measure

$$(p * q)(A) = \int_{x \in G} \int_{y \in G} \chi_A(x * y) \ dq(y) \ dp(x),$$

where $\chi_A$ denotes the characteristic function of $A$.

Equipped with the operation $*$ defined in this way, $\beta S$ becomes a right topological semigroup. This means that for each $p \in \beta S$, the right-translation mapping $\rho_p : \beta G \longrightarrow \beta G$ given by $\rho_p(q) = q * p$ is continuous, although the left-translation mappings $\lambda_p : \beta G \longrightarrow \beta G$ ($\lambda_p(q) = p * q$) are not continuous in general. It is important to note that, even when $(S, *)$ is a commutative semigroup, the extended operation $*$ is in general not commutative, and nonprincipal ultrafilters $p \in S^*$ do not have inverses even when $(S, *)$ is a group. However, whenever $S, T$ are semigroups and $f : S \longrightarrow T$ is a semigroup homomorphism (this is, $(\forall x, y \in S)(f(x * y) = f(x) * f(y)))$ then so is $\beta f : \beta S \longrightarrow \beta T$, which means that $f(p * q) = f(p) * f(q)$ for all $p, q \in \beta S$. In most cases (the exact hypotheses needed to ensure this are stated in [21, Theorem 4.28], and are satisfied by all semigroups considered here) the Čech-Stone remainder $S^*$ is a subsemigroup of $\beta S$. Being (always) also a closed subset of the compact space $\beta S$, we conclude that, in the cases that we are concerned with here, $S^*$ is itself a compact right-topological semigroup.

Now we turn to another useful notion that happens to be quite closely related to ultrafilters. Let $X$ be a set and suppose that we have a family $\mathscr{A} \subseteq \mathfrak{P}(X)$ of subsets of $X$. Then we say that $\mathscr{A}$ is **partition regular**, or a **coideal**, if $\mathscr{A}$ is closed under supersets and, whenever an element of $\mathscr{A}$ is partitioned into two cells, the family $\mathscr{A}$ necessarily contains at least one of the cells. One of the main reasons that this notion is relevant for our study of ultrafilters, is that the following three conditions are equivalent for a family $\mathscr{A} \subseteq \mathfrak{P}(X)$ of subsets of $X$.

(i) $\mathscr{A}$ is partition regular,

(ii) there exists an ultrafilter $p$ all of whose elements belong to the family $\mathscr{A}$ (this is, $p \subseteq \mathscr{A}$); and

(iii) whenever $\mathscr{F}$ is a filter all of whose elements belong to the family $\mathscr{A}$ (i.e. $\mathscr{F} \subseteq \mathscr{A}$), then there exists an ultrafilter $p$ extending $\mathscr{F}$ all of whose elements belong to the family $\mathscr{A}$ (this is, $\mathscr{F} \subseteq p \subseteq \mathscr{A}$).

## 1.3 Finite Products, Strongly Productive Ultrafilters

Let $(S, *)$ be a semigroup and $\alpha$ an ordinal. Whenever we have a sequence $\vec{x} = \langle x_\xi \big| \xi < \alpha\rangle$ of elements of $S$, we will define the **set of finite products of** $\vec{x}$ as

$$\mathrm{FP}(\vec{x}) = \left\{ \prod_{\xi\in a} x_\xi \middle| a \in [\alpha]^{<\omega} \setminus \{\varnothing\} \right\},$$

where the products are computed in increasing order of indices (i.e. if $a = \{\xi_0, \dots, \xi_n\}$ with $\xi_0 < \xi_1 < \cdots < \xi_n$ then $\prod_{\xi\in a} x_\xi = \xi_0 * \xi_1 * \cdots * \xi_n$). The ordinal index $\alpha$ of our sequences will typically be at most $\omega$. When we have an $\alpha$-sequence $\vec{x}$ of elements of a semigroup $S$, and $\beta < \alpha$, we will also use the notation $\mathrm{FP}_\beta(\vec{x})$ to denote the set $\mathrm{FP}(\vec{y})$ where $\vec{y}$ is the $\beta$-sequence given by $y_\gamma = x_{\beta+\gamma}$. If we allow a certain degree of informality and language abuse, we can simply think that

$$\mathrm{FP}_\beta(\vec{x}) = \mathrm{FP}(\langle x_\gamma \big| \beta \leq \gamma < \alpha\rangle).$$

**Definition 1.1.** An ultrafilter $p \in \beta S$ will be called **strongly productive** if for every $A \in p$ there exists an $\omega$-sequence $\vec{x}$ of elements of $S$ such that $p \ni \mathrm{FP}(\vec{x}) \subseteq A$.

It is not hard to see that elements $x \in S$ are strongly productive if and only if they are idempotent (i.e. $x * x = x$). We will see in Chapter 2 that, for a good amount of semigroups $S$, nonprincipal strongly productive ultrafilters $p \in \beta S$ must also satisfy the equation $p * p = p$, i.e. they are idempotents as well.

If our semigroup is abelian, we will typically use "additive notation", meaning that we will denote the semigroup operation by $+$ (hence our semigroup will now be

$(S,+)$). Accordingly, we will also write $\mathrm{FS}(\vec{x})$ instead of $\mathrm{FP}(\vec{x})$ ("finite sums" rather than "finite products"); and notice that, in this case, we can forget all requirements having to do with the order in which we add the elements of the sequence, so it is possible to "rearrange" a given sequence $\vec{x}$ and still get the same FS-set. This means that, if $\vec{x}$ and $\vec{y}$ are two sequences (not necessarily indexed by the same ordinal) with the same range and such that, for every $z$ in the common range, the cardinality of the fiber of $z$ is the same with respect to both sequences (this is, if $|\{\alpha < \mathrm{dom}(\vec{x}) \big| x_\alpha = z\}| = |\{\alpha < \mathrm{dom}(\vec{y}) \big| y_\alpha = z\}|$), then $\mathrm{FS}(\vec{x}) = \mathrm{FS}(\vec{y})$. Hence it makes sense to define, for every $X \subseteq S$, $\mathrm{FS}(X)$ to be the set $\mathrm{FS}(\vec{x})$ where $\vec{x}$ is any injective sequence with range $X$. In this context, we state the following definition.

**Definition 1.2.** A **strongly summable ultrafilter** is just a strongly productive ultrafilter on an additively denoted commutative semigroup $(S,+)$. This is, $p \in \beta S$ is strongly summable if for every $A \in p$ there exists an $\omega$-sequence $\vec{x}$ of elements of $S$ such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A$.

The concept of strongly summable ultrafilter was first introduced in the case of the additive semigroup of positive integers $(\mathbb{N},+)$ in [17, Definition 2.1] by Hindman upon suggestion of van Douwen (cf. also the notes at the end of [21, Chapter 12]), although Hindman had already (inadvertently) proved in [15, Theorem 3.3] that the existence of strongly summable ultrafilters on $\mathbb{N}$ follows from $\mathsf{CH}$. They were also considered (independently) by Matet [31].

From now on, we will assume that all of our semigroups have an identity, which we denote by $e$ when the semigroup is multiplicatively denoted and by 0 when it is additively denoted. When dealing with strongly productive (or strongly summable, accordingly) ultrafilters, it will of course be of crucial importance to determine when two of them "look the same", in a sense that takes into account the algebraic structure of the FP-sets involved. The relevant definition was introduced in [6, p. 84]. Before we present it, we need to talk about a further notion, which stems from the fact that, when dealing with sets of the form $\mathrm{FP}(\vec{x})$, if each finite product from this set can be expressed uniquely as such then the situation is much more comfortable. To simplify notation, we make the convention that for any $\alpha$-sequence $\vec{x}$ of elements of some semigroup $S$, the **empty product** (respectively **empty sum**) equals the identity element:

$$\prod_{n\in\varnothing} x_n = e \qquad\qquad (\text{respectively } \sum_{n\in\varnothing} x_n = 0).$$

**Definition 1.3.** An $\alpha$-sequence $\vec{x}$ of elements of a semigroup $S$ is said to satisfy **uniqueness of products** (respectively **uniqueness of sums**) if whenever $a, b \in [\alpha]^{<\omega}$ are such that

$$\prod_{n\in a} x_n = \prod_{n\in b} x_n \qquad\qquad (\text{respectively } \sum_{n\in a} x_n = \sum_{n\in b} x_n),$$

it must be the case that $a = b$.

In particular, if $\vec{x}$ satisfies uniqueness of products (respectively uniqueness of sums) then $\vec{x}$ is injective, and $e \notin \mathrm{FP}(\vec{x})$ (respectively $0 \notin \mathrm{FS}(\vec{x})$). We now finally have under our belt the appropriate terminology needed to express our very particular notion of isomorphism.

**Definition 1.4.** Let $S, T$ be semigroups and $p \in \beta S$, $q \in \beta T$ be strongly productive ultrafilters. We say that $p$ and $q$ are **multiplicatively isomorphic** if there are sequences $\vec{x} \in {}^{\omega}S$, $\vec{y} \in {}^{\omega}T$ satisfying uniqueness of products, such that $\mathrm{FP}(\vec{x}) \in p$, $\mathrm{FP}(\vec{y}) \in q$, and the mapping $\varphi : \mathrm{FP}(\vec{x}) \longmapsto \mathrm{FP}(\vec{y})$ given by $\varphi(\prod_{n\in a} x_n) = \prod_{n\in a} y_n$ sends $p$ to $q$ (this is, $\varphi(p) = q$). We say that the mapping $\varphi$ is a **multiplicative isomorphism**.

(If either of the semigroups $S$ or $T$ is additively denoted, replace "products" by "sums", FP by FS, and so on, accordingly. If *both* semigroups $S$ and $T$ are additively denoted then we talk about **additively isomorphic** and **additive isomorphism**.)

Thus, the notion of a multiplicative (respectively additive) isomorphism is a very natural strengthening of the notion of Rudin-Keisler equivalence, which incorporates the fact that we are interested in the algebraic structure of FP- (respectively FS-) sets.

Next, we mention a concept that will be useful in several places of this work.

**Definition 1.5.** Given a semigroup $S$ and a sequence $\vec{x}$ (indexed by some ordinal $\alpha$) of elements from $S$ that satisfies uniqueness of sums, we say that the sequence $\vec{y}$

(indexed by some ordinal $\delta$) is a **product subsystem** of the sequence $\vec{x}$ if $\mathrm{FP}(\vec{y}) \subseteq \mathrm{FP}(\vec{x})$ and, if for every $\beta < \delta$ we let $a_\beta \in [\alpha]^{<\omega}$ be such that $y_\beta = \prod_{\xi \in a_\beta} x_\xi$, then $\beta < \gamma < \delta$ implies that $\max(a_\beta) < \min(a_\gamma)$. If the semigroup is additively denoted then we will say that $\vec{y}$ is a **sum subsystem**.

We finish this section by mentioning a significant lemma which will be of use throughout the rest of this thesis.

**Lemma 1.6.** *Let $S$ be a semigroup, and let $\vec{x}$ be an $\omega$-sequence of elements of $S$. Then the set* $\mathrm{FP}_\infty(\vec{x}) = \{p \in \beta S \big| (\forall n < \omega)(\mathrm{FP}_n(\vec{x}) \in p)\}$ *is a nonempty closed subsemigroup of $\beta S$. Moreover, if the sequence $\vec{x}$ satisfies uniqueness of products, then* $\mathrm{FP}_\infty(\vec{x}) \subseteq S^*$.

*Proof.* It is clear that the family $\{\mathrm{FP}_n(\vec{x}) \big| n < \omega\}$ is a filter base, thus $\mathrm{FP}_\infty(\vec{x})$ is a closed subset of $\beta S$. To see that it is also a subsemigroup, we let $p, q \in \mathrm{FP}_\infty(\vec{x})$ and aim to prove that, for every $n < \omega$, $\mathrm{FP}_n(\vec{x}) \in p * q$. Since $\mathrm{FP}_n(\vec{x}) \in p$, it suffices to show that, for every $x \in \mathrm{FP}_n(\vec{x})$, the set $\{y \in S \big| x * y \in \mathrm{FP}_n(\vec{x})\} \in q$. So let $x \in \mathrm{FP}_n(\vec{x})$ and assume that $x = \prod_{i \in a} x_i$ ($a \subseteq \omega$, $n \leq \min(a)$). Let $m = \max(a)+1$, then $\mathrm{FP}_m(\vec{x}) \in q$, so it suffices to show that, for every $y \in \mathrm{FP}_m(\vec{x})$, $x * y \in \mathrm{FP}_n(\vec{x})$. So let $y \in \mathrm{FP}_m(\vec{x})$ and assume that $y = \prod_{j \in b} x_j$ ($b \subseteq \omega$, $min(b) \geq m$). Then we have that $\max(a) < \min(b)$, thus

$$x * y = \left(\prod_{i \in a} x_i\right) * \left(\prod_{j \in b} x_j\right) = \prod_{k \in a \cup b} x_k \in \mathrm{FP}_n(\vec{x})$$

and we are done.

The "moreover" part follows quite easily from the main statement of the lemma, for if $\vec{x}$ satisfies uniqueness of products then the filter generated by the sets $\mathrm{FP}_n(\vec{x})$ is free (meaning its intersection is empty). $\square$

**Corollary 1.7.** *Let $S$ be a semigroup, and let $\{\vec{x_\alpha}\big|\alpha < \kappa\}$ be a family of $\omega$-sequences of elements of $S$. Then the set*

$$\{p \in \beta S \big| \{\mathrm{FP}_n(\vec{x_\alpha})\big|\alpha < \kappa \wedge n < \omega\} \subseteq p\}$$

*is a closed subsemigroup of $\beta S$.*

## 1.4 The Boolean Group

Recall that a Boolean Group is a group that contains only (except for the identity) elements of order 2. It is well-known that every Boolean group is abelian and that, in fact, for every infinite cardinal $\kappa$ there is a unique (up to isomorphism) Boolean group of cardinality $\kappa$ (meanwhile, for a finite cardinal $n$, there exists a Boolean group of cardinality $n$ if and only if $n$ is a power of two, in which case this Boolean group is unique up to isomorphism). Throughout this work, we will denote the Boolean group of cardinality $\kappa$ by $\mathbb{B}_\kappa$. Our favourite realization for this group – our favourite way of thinking of it– is as the collection of finite subsets of $\kappa$ with the symmetric difference as group operation $([\kappa]^{<\omega}, \triangle)$. Sometimes we will also encounter the group $\mathbb{B}_\kappa$ realized as the subgroup

$$\left\{x \in \bigoplus_{\alpha<\kappa} \mathbb{T} \middle| (\forall \alpha < \kappa) \left(\pi_\alpha(x) \in \left\{0, \frac{1}{2}\right\}\right)\right\}$$

of the direct sum $\bigoplus_{\alpha<\kappa} \mathbb{T}$ of $\kappa$ copies of $\mathbb{T}$.

We pay particular attention to the Boolean group of cardinality $\omega$. From now on, the words "the Boolean group" (without reference to any cardinality) will always refer to this group, which we denote by $\mathbb{B}$ (without a cardinal in the subindex). Usually we will think of this group as being just $([\omega]^{<\omega}, \triangle)$. One of the main offshots of this dissertation work is that strongly summable ultrafilters on an arbitrary abelian group $G$ (and also many strongly productive ultrafilters in several important noncommutative semigroups), in a sense, essentially look like strongly summable ultrafilters on $\mathbb{B}$. More concretely, one of the main results from Chapter 3 (namely, Corollary 3.28) will show that one does not really lose generality by considering only strongly summable ultrafilters on $\mathbb{B}$.

When dealing with the Boolean group and considering sets of finite sums, we will write $\mathrm{F}\triangle(X)$ instead of $\mathrm{FS}(X)$ to emphasize that the group operation is the symmetric difference. Notice that the fact that every element has order 2 implies that for any sequence $\vec{x}$, $\mathrm{F}\triangle(\vec{x})$ is essentially equal to $\mathrm{F}\triangle(\mathrm{ran}(\vec{x}))$. By this we mean that $\mathrm{F}\triangle(\vec{x})$ is equal to either $\mathrm{F}\triangle(\mathrm{ran}(\vec{x}))$ or $\mathrm{F}\triangle(\mathrm{ran}(\vec{x})) \cup \{0\}$ (the latter case can happen, for example, if $\vec{x}$ is not injective). Hence a nonprincipal ultrafilter $p \in \mathbb{B}^*$ is strongly summable if and only if for every $A \in p$ there is an infinite set $X \subseteq \mathbb{B}$

such that $p \ni \mathrm{F}\triangle(X) \subseteq A$.

We will use the fact that $\mathbb{B}_\kappa$ has a structure of $\kappa$-dimensional vector space over the field with two elements $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ (scalar multiplication being the obvious one). Note that for $X \subseteq \mathbb{B}_\kappa$, the subspace spanned by $X$, which is the same as the subgroup generated by $X$, is exactly $\mathrm{F}\triangle(X) \cup \{\varnothing\}$, because nontrivial linear combinations (i.e. linear combinations in which not all scalars equal zero) of elements of $X$ are exactly finite sums (or symmetric differences) of elements of $X$. We can use this to figure out how do subsets $X \subseteq \mathbb{B}_\kappa$ satisfying uniqueness of sums look like.

**Proposition 1.8.** *For $X \subseteq \mathbb{B}$, the following are equivalent:*

*(i) $X$ satisfies uniqueness of finite sums.*

*(ii) $\varnothing \notin \mathrm{F}\triangle(X)$.*

*(iii) $X$ is linearly independent.*

*Proof.* Given the observation from the previous paragraph relating finite sums and nontrivial linear combinations, it is straightforward to see that (ii) is equivalent to (iii). That either of these is equivalent to (i) is as follows. If $X$ does not satisfy uniqueness of finite sums, then there are two distinct nonempty $A, B \in [X]^{<\omega}$ such that $\underset{x \in A}{\triangle} x = \underset{x \in B}{\triangle} x$, whereby adding $\underset{x \in B}{\triangle} x$ to both sides of this equation yields

$\bigtriangleup_{x\in A\triangle B} x = \varnothing$, and since $A \neq B$ this sum is nonempty, i.e. $A \triangle B \neq \varnothing$, so that $\varnothing \in \mathrm{F}\triangle(X)$. Conversely, if $\varnothing \in \mathrm{F}\triangle(X)$, i.e. if there is a nonempty $A$ such that $\bigtriangleup_{x\in A} x = \varnothing$, then by picking any nonempty $B$ disjoint from $A$ we get that $B \neq B \cup A$ but nevertheless

$$\bigtriangleup_{x\in B\cup A} x = \left(\bigtriangleup_{x\in B} x\right) \triangle \left(\bigtriangleup_{x\in A} x\right) = \bigtriangleup_{x\in B} x.$$

□

Thus when we have a set $\mathrm{F}\triangle(Y)$ such that $Y$ is not linearly independent, we can always choose a basis $X$ for the subspace $\mathrm{F}\triangle(Y)$ spanned by $Y$, and we will have that $\mathrm{F}\triangle(X) = \mathrm{F}\triangle(Y) \setminus \{\varnothing\}$. This means that, when considering sets of the form $\mathrm{F}\triangle(X)$, we can assume without loss of generality that $X$ is linearly independent. Another way to see this is the following: let $p \in B^*$ be a strongly summable ultrafilter, and let $A \in p$. Since $p$ is nonprincipal, $\{\varnothing\} \notin p$ and hence $A \setminus \{\varnothing\} \in p$. Therefore we can choose an $X$ such that $p \ni \mathrm{F}\triangle(X) \subseteq A \setminus \{0\}$, so $\mathrm{F}\triangle(X) \subseteq A$ and $X$ must be linearly independent.

We need to also introduce the notion of support.

**Definition 1.9.** For a linearly independent set $X \subseteq \mathbb{B}$, we define for an element $y \in \mathrm{F}\triangle(X)$ the $X$**-support** of $y$, denoted by $\mathrm{supp}_X(y)$, as the (unique, by linear independence of $X$) finite set of elements of $X$ whose sum equals $y$. This is,

$$y = \bigtriangleup_{x\in \mathrm{supp}_X(y)} x.$$

If $Y \subseteq \mathrm{F}\triangle(X)$ then, we also define the $X$-support of $Y$ as

$$\mathrm{supp}_X(Y) = \bigcup_{y \in Y} \mathrm{supp}_X(y).$$

Similarly, we define the $X$-support of a sequence of elements of $\mathrm{F}\triangle(X)$ as the $X$-support of its range.

It will be convenient to stipulate the convention that $\mathrm{supp}_X(\varnothing) = \varnothing$. Then it is readily checked that the function $\mathrm{supp}_X : \mathrm{F}\triangle(X) \cup \{\varnothing\} \longrightarrow ([X]^{<\omega}, \triangle)$ is a group isomorphism (in fact, a linear transformation between the two vector spaces), in other words, $\mathrm{supp}_X(x \triangle y) = \mathrm{supp}_X(x) \triangle \mathrm{supp}_X(y)$ for all $x, y \in \mathrm{F}\triangle(X)$; and more generally $\mathrm{supp}_X\left(\bigtriangleup_{x \in A} x\right) = \bigtriangleup_{x \in A} \mathrm{supp}_X(x)$ for all $A \in [\mathrm{F}\triangle(X)]^{<\omega}$. This is the really crucial feature of the $X$-support, and it will be used ubiquitously throughout this work.

When dealing with the extension of the operation $\triangle$ to all of $\beta\mathbb{B}$, in order to avoid confusion we will use the symbol $\blacktriangle$ to denote the aforementioned extension. We will also use that symbol to denote translates of sets, $x \blacktriangle A = \{x \triangle y \big| y \in A\}$. Thus, with this notation,

$$p \blacktriangle q = \{A \subseteq \mathbb{B} \big| \{x \in \mathbb{B} \big| x \blacktriangle A \in q\} \in p\}$$

for any two $p, q \in \beta\mathbb{B}$.

We will now present some important notions first introduced by Blass in [3, p. 92] (an article that appeared in the same volume as that of Hindman's [17] where strongly summable ultrafilters are first defined). Ever since their inception, strongly summable ultrafilters have always been inextricably related to the notions that we will present in what follows.

**Definition 1.10.**

(i) A family $X \subseteq \mathbb{B}$ is said to be **disjoint** if its elements are pairwise disjoint (as anyone would expect). It is said to be **ordered** or **in block position** if for every two distinct $x, y \in X$, either $\max(x) < \min(y)$ or $\max(y) < \min(x)$.

(ii) An ultrafilter $p \in \mathbb{B}^*$ is said to be a **union ultrafilter** if for every $A \in p$ there exists a disjoint family $X$ such that $p \ni \mathrm{F}\triangle(X) \subseteq A$, and $p$ is an **ordered union ultrafilter** if for every $A \in p$ there exists an ordered family $X$ such that $p \ni \mathrm{F}\triangle(X) \subseteq A$. Thus, notice that union and ordered union ultrafilters are particular cases of strongly summable ultrafilters on $\mathbb{B}$.

Notice that the only idempotent of $\mathbb{B}$, namely $0 = \varnothing$, is a strongly summable ultrafilter but not a union ultrafilter. For nonprincipal strongly summable ultrafilters on $\mathbb{B}$, however, it is not obvious that the two notions are distinct. We will show in Chapter 4 that it is consistent that there exists a nonprincipal strongly summable ultrafilter on $\mathbb{B}$ which is not a union ultrafilter (in fact, this ultrafilter is not even additively isomorphic to any union ultrafilter).

The reason that union ultrafilters are so important in this realm is that significantly many strongly productive ultrafilters are multiplicatively isomorphic to union ultrafilters (but not all, as we remarked in the previous paragraph). We will see next that when this is the case, the multiplicative isomorphism witnessing this fact is actually very simple.

**Proposition 1.11.** *Let $p$ be a strongly productive ultrafilter on some semigroup $S$. If $p$ is multiplicatively isomorphic to a union ultrafilter, and this is witnessed by the mapping $\prod_{n\in a} x_n \longmapsto \bigcup_{n\in a} y_n$ from $\mathrm{FP}(\vec{x})$ to $\mathrm{F}\triangle(Y)$, then the mapping $\psi : \mathrm{FP}(\vec{x}) \longrightarrow [\omega]^{<\omega}$ given by $\psi(\prod_{n\in a} x_n) = a$ also maps $p$ to a union ultrafilter.*

*Proof.* We only need to show that for any union ultrafilter $q$ and any pairwise disjoint $Y = \{y_n \big| n < \omega\}$ such that $\mathrm{F}\triangle(Y) \in q$, the mapping $\varphi$ given by $\bigcup_{n\in a} y_n \longmapsto a$ maps $q$ to another union ultrafilter. Once we prove this, then given the hypothesis of the theorem we can just compose the mapping $\varphi$ with the original isomorphism to get the $\psi$ that we need. So let $r$ be the image of $q$ under such mapping, and let $A \in r$. Then since $B = \varphi^{-1}[A] \in q$, there is a pairwise disjoint $X$ such that $q \ni \mathrm{F}\triangle(X) \subseteq B \cap \mathrm{F}\triangle(Y)$. Since $X$ is pairwise disjoint and contained in $\mathrm{F}\triangle(Y)$, it is readily checked that for distinct $x, w \in X$, if $x = \bigcup_{n\in a} y_n$ and $w = \bigcup_{n\in b} y_n$ then $a \cap b = \varnothing$. Hence the family $Z = \{a \in [\omega]^{<\omega} \big| \bigcup_{n\in a} y_n \in X\}$ is pairwise disjoint. Note moreover that all finite unions are preserved in the sense that, for $x_0, \dots, x_n \in X$ such that $x_i = \bigcup_{k\in a_i} y_k$, we have that $\bigcup_{i=0}^{n} x_i = \bigcup_{k\in a} y_k$ where $a = \bigcup_{i=0}^{n} a_i$, i.e. $\varphi\left(\bigcup_{i=0}^{n} x_i\right) =$

$\bigcup_{i=0}^{n} \varphi(x_i)$. This means that $\varphi[\mathrm{F}\triangle(X)] = \mathrm{F}\triangle(Z)$, thus $r \ni \mathrm{F}\triangle(Z) \subseteq A$ and we are done. $\square$

# 2 Strongly Productive Ultrafilters: Basic Properties

This chapter outlines some basic results that deal with strongly productive/summable ultrafilters. It should be mentioned that all of the results from Sections 2.3, 2.4 and 2.5 arose from a joint work with Martino Lupini, and they can be found in greater generality in the paper [13].

## 2.1 Idempotent ultrafilters

Of special importance for the present study are idempotent ultrafilters on semigroups. In a sense, these objects have been historically intertwined with strongly productive ultrafilters. An idempotent ultrafilter on a semigroup $S$ is just an ultrafilter satisfying the equation $p = p * p$. If we recall that the Čech-Stone compactification $\beta S$ of $S$ is a right-topological compact semigroup, then the following classical result, whose proof we include just for convenience of the reader, implies the existence of idempotent ultrafilters. Moreover, since the Čech-Stone remainder

$S^* = \beta S \setminus S$ is a closed subsemigroup of $\beta S$, we ensure the existence of nonprincipal idempotent ultrafilters as well.

**Lemma 2.1** (Ellis)**.** *Every compact right-topological semigroup has idempotents.*

*Proof.* Let $S$ be a compact right-topological semigroup. Let $\mathbb{P}$ be the preorder whose conditions are nonempty compact subsemigroups of $S$, with $T' \leq T \iff T' \subseteq T$. Then every chain $\mathcal{C} \subseteq \mathbb{P}$ has the finite intersection property, whereby compactness of $S$ ensures that $\bigcap \mathcal{C}$ (which is, of course, compact as well) is nonempty. Since it is also clearly a subsemigroup of $S$, we conclude that $\bigcap \mathcal{C} \in \mathbb{P}$ is a lower bound for $\mathcal{C}$, hence Zorn's Lemma gives us a minimal element $T \in \mathbb{P}$.

Grab any $x \in T$ (at the end of the argument we will actually have shown that $T = \{x\}$), and consider the shift, or translate, $T*x$ of $T$ by $x$. This is the continouos image of a compact set, hence compact, and it is very easy to check that it is a (clearly nonempty) subsemigroup of $S$. Thus $T * x \in \mathbb{P}$, while at the same time $T * x \subseteq T$, so by minimality of $T$ we must have that $T = T * x$. In particular, there is a $y \in T$ such that $y * x = x$.

The previous paragraph ensures that the following set is nonempty:

$$T' = \{y \in T \big| y * x = x\} = \rho_x^{-1}[\{x\}].$$

Now $T'$ is the continuous preimage of a closed set, hence closed, hence compact. And it is not at all hard to check that it is a subsemigroup. Thus $T' \in \mathbb{P}$, and

$T' \subseteq T$ so by minimality of $T$ we get that $T = T'$ and, in particular (since $x \in T$), we get $x * x = x$ so $x$ is the idempotent that testifies for our statement. □

Next, we aim to prove Hindman's finite sums theorem. Since we want to prove it in full generality, it will be phrased in the form of finite products on a semigroup. We introduce some useful notation: If $p \in \beta S$ is an ultrafilter, and $A \subseteq S$, we let $A^\star(p) = \{x \in A \big| x^{-1} * A \in p\}$. We will drop $p$ from the notation when it is clear from the context. The really important property about this operation, is that if $p$ is an idempotent and $A \in p$, then $A^\star \in p$ and $(A^\star)^\star = A^\star$ [21, Lemma 4.14].

The following proof will be useful as a "template" when we try, later on, to prove the existence of certain kinds of ultrafilters.

**Theorem 2.2** (Hindman)**.** *Let $\vec{x}$ be an $\omega$-sequence of elements of a semigroup $S$ with an infinite range, and suppose that we have a partition* $\mathrm{FP}(\vec{x}) = A_0 \cup A_1$. *Then there exists an $i \in 2$ and an injective $\omega$-sequence $\vec{y}$ such that* $\mathrm{FP}(\vec{y}) \subseteq A_i$. *In particular, whenever we partition an infinite semigroup into finitely many pieces, one of the pieces must contain an* FP*-set.*

*Proof.* Use the Ellis-Numakura lemma together with Lemma 1.6 to find an idempotent nonprincipal ultrafilter $p = p * p \in \mathrm{FP}_\infty(\vec{x}) \cap S^*$. Let $i \in 2$ be such that $A_i \in p$. We will inductively construct the sequence $\vec{y}$, but instead of aiming for $\mathrm{FP}(\vec{x})$ to be a subset of $A_i$, we will make it a subset of $B_0 = A_i^\star$. First we notice that, if we had chosen the sequence $\vec{y}$ already, we would have the equality $\mathrm{FP}(\vec{y}) = y_0 * \mathrm{FP}_1(\vec{y}) \cup \mathrm{FP}_1(\vec{y})$. Thus it is sufficient to require that $y_0 \in B_0$, as long as we also have that $\mathrm{FP}_1(\vec{y}) \subseteq B_0 \cap y_0^{-1} * B_0$. This transfers the problem of choosing the sequence $\vec{y}$ in such a way that $\mathrm{FP}(\vec{y}) \subseteq B_0$ to the problem of choosing the tail end of the sequence, $\langle y_n \big| n \geq 1\rangle$, in such a way that $\mathrm{FP}_1(\vec{y}) = \mathrm{FP}(\langle y_n \big| n \geq 1\rangle \subseteq B_0 \cap y_0^{-1} * B_0$, which we will achieve by transferring again the problem: choosing $y_1 \in B_1 = (B_0 \cap y_0^{-1} * B_0)^\star$ and making sure that $\mathrm{FP}_2(\vec{y}) = \mathrm{FP}(\langle y_n \big| n \geq 2\rangle) \subseteq B_1 \cap y_1^{-1} * B_1$ will do. As can be expected by now, we transfer again the problem, meaning that it suffices to choose $y_2 \in B_2 = (B_1 \cap y_1^{-1} * B_1)^\star$ and to make sure that $\mathrm{FP}_3(\vec{y}) = \mathrm{FP}(\langle y_n \big| n \geq 3\rangle \subseteq B_2 \cap y_2^{-1} * B_2$, which we will ensure by picking $y_3 \in B_3 = (B_2 \cap y_2^{-1} * B_2)^\star$ and making sure that $\mathrm{FP}_4(\vec{y}) = \mathrm{FP}(\langle y_n \big| n \geq 4\rangle) \subseteq B_3 \cap y_3^{-1} * B_3$, and so on; this process will in the end (after $\omega$ many steps) yield the desired sequence $\vec{y}$.

For the "in particular" claim, just grab your favourite countably infinite subset of $S$ and arrange it in an $\omega$-sequence $\vec{x}$, and let the given partition of $S$ induce a corresponding partition on $\mathrm{FP}(\vec{x})$. □

Note that, if we additionally assume that $\vec{x}$ satisfies the uniqueness of sums, then it is easy to slightly modify the proof of the previous theorem in such a way that the chosen sequence $\vec{y}$ is a product subsystem of the sequence $\vec{x}$. All one needs to do is ensure that, at the $n$-th step, we choose $y_n$ to be an element of

$B_n \setminus \mathrm{FP}(\langle x_k \big| k \leq m \rangle)$ rather than $B_n$, where $m = \max(\bigcup_{i<n} a_i)$ if we assume that $y_i = \prod_{j \in a_i} x_j$ for all $i < n$ (this set is still an element of $p$, since we were assuming that $p$ was nonprincipal).

As a particular case, we get Hindman's theorem, which was originally proved [16, Theorem 3.1] by combinatorial methods. For a short combinatorial proof see [2].

**Corollary 2.3** (Hindman)**.** *Whenever we partition* $\mathbb{N} = A_0 \cup A_1$*, there exists an* $i \in 2$ *and an injective* $\omega$*-sequence* $\vec{x}$ *such that* $\mathrm{FS}(\vec{x}) \subseteq A_i$*.*

*Proof.* In the previous theorem, just let $\mathbb{N} = S$ (or let $\vec{x}$ be given by $x_n = 2^n$ and notice that in this case $\mathbb{N} = \mathrm{FS}(\vec{x})$). □

This motivates the following definition.

**Definition 2.4.** An ultrafilter $p$ on a semigroup $S$ is called **weakly productive** (or **weakly summable** if the semigroup is additively denoted) if for every $A \in p$ there exists an injective $\omega$-sequence of elements of $S$ such that $\mathrm{FP}(\vec{x}) \subseteq A$.

The proof of Hindman's theorem actually shows that idempotent ultrafilters are weakly productive. In fact, it is easy to argue that being weakly productive is equivalent to belonging to the closure in $\beta S$ of the set of all idempotents, for a set $A \subseteq S$ contains an FP-set if and only if its closure $\bar{A}$ in $\beta S$ contains an idempotent (one of the implications on this statement is given by Hindman's theorem, whilst the other is Lemma 1.6).

## 2.2 Existence (Consistently)

We are aiming to prove that, if $S$ is a semigroup, then it is consistent that there exists a strongly productive (or summable) ultrafilter on $S$. We were not able to get a statement in such great generality (in particular, for all we know it is possible that in ZFC there is a semigroup without strongly summable ultrafilters), but we have been able to isolate a reasonably broad class of semigroups for which the result holds. Along the way, we will also see that for some semigroups the existence of strongly summable ultrafilters can actually be proved in ZFC (in fact we will look at two examples of semigroups on which every ultrafilter is strongly summable).

**Definition 2.5.** Let $S$ be a semigroup. We say that $S$ is

(i) **left cancellative** if whenever $x * y = x * z$ $(x, y, z \in S)$ we have that $y = z$.

(ii) **right cancellative** if whenever $y * x = z * x$ $(x, y, z \in S)$ we have that $y = z$.

(iii) **weakly left cancellative** if for every $u, v \in S$ there are only finitely many $x \in S$ such that $u * x = v$.

(iv) **weakly right cancellative** if for every $u, v \in S$ there are only finitely many $x \in S$ such that $x * u = v$.

These definitions will allow us to delineate the class of semigroups $S$ for which we can show, under certain set-theoretic hypotheses, that strongly productive ultrafilters exist on $S$. We first note, however, that [21, Theorem 6.35] establishes that if $S$ is right cancellative and weakly left cancellative, then $S^* * S^*$ (this is, the collection of all products of two nonprincipal ultrafilters) is nowhere dense in $S^*$. It will be shown in Sections 2.3, 2.4 and 2.5 that, for a large class of semigroups $S$ (all of which are right cancellative and weakly left cancellative), every strongly productive ultrafilter $p$ on $S$ is idempotent, in particular $p = p * p \in S^* * S^*$. Thus for these semigroups, there are lots (from a topological perspective) of ultrafilters that are *not* strongly productive (the set of such ultrafilters is dense in $S^*$).

**Example 2.6.** Consider the semigroup $(\mathbb{N}, *)$ where $n * m = \min\{n, m\}$ for all $n, m \in \mathbb{N}$. Then given any finite $a \subseteq \mathbb{N}$, we have that $\prod_{n \in a} n = \min(a)$. Hence if $p \in \beta N$ is any ultrafilter and $A \in p$, letting $\vec{x} = \langle x_n \big| n < \omega \rangle$ be any enumeration of the set $A$ we get that $p \ni \mathrm{FP}(\vec{x}) \subseteq A$, because $\mathrm{FP}(\vec{x}) = A$. Thus every ultrafilter on this semigroup is strongly productive. Note that this semigroup is neither weakly left cancellative nor right cancellative. In an entirely analogous way it is possible to prove that every ultrafilter on the semigroup $(\mathbb{N}, \max)$ is strongly productive. Note that this semigroup is weakly left cancellative but not right cancellative (although it is weakly right cancellative). Thus the conditions mentioned in the previous paragraph seem to be necessary to ensure that some ultrafilters will not be strongly summable. Note that neither of these two semigroups have sequences of elements with the uniqueness of sums, which shows that the hypotheses from the following proposition (which is a particular case of [21, Lemma 6.31]) are also necessary.

**Proposition 2.7.** *Let $S$ be an infinite semigroup that is right cancellative and weakly left cancellative. Then there exists a sequence $\vec{x} \in {}^{\omega}S$ satisfying uniqueness of products.*

*Proof.* We will recursively construct the terms $x_n$ of the sequence. We let $x_0$ be any element of $S \setminus \{e\}$. Now assume that we have already chosen a partial sequence $\langle x_k \big| k < n\rangle$ which satisfies the uniqueness of products, this is, the elements $\prod_{i\in a} x_i$ for $\varnothing \neq a \subseteq n$ are $2^n - 1$ pairwise distinct elements of $S$. Since $S$ is weakly left cancellative, for each fixed $a, b \subseteq n$, $a \neq \varnothing \neq b$, there are only finitely many solutions to the equation (on the variable $x$)

$$\left(\prod_{i\in a} x_i\right) * x = \prod_{j\in b} x_j$$

Thus (since $S$ is infinite) it is possible to choose an $x_n$ which is not a solution of any of those equations for any $a, b$. This means that the set $\mathrm{FP}(\langle x_k \big| k < n\rangle) * x_n$ is disjoint from $\mathrm{FP}(\langle x_k \big| k < n\rangle)$. Moreover, for any two distinct $a, b \subseteq n$ $(a \neq \varnothing \neq b)$ we have that $\left(\prod_{i\in a} x_i\right) * x_n \neq \left(\prod_{j\in b} x_j\right) * x_n$ because $S$ is right cancellative and $\langle x_k \big| k < n\rangle$ satisfies uniqueness of products. All of this together means that the new (slightly longer) sequence $\langle x_k \big| k < n\rangle \frown \langle x_n\rangle = \langle x_k \big| k \leq n\rangle$ satisfies uniqueness of products, so the induction can continue and we are done. □

We will now proceed to prove our existence result.

**Theorem 2.8.** *Assume that* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*, and let* $S$ *be an infinite semigroup such that there exists a sequence* $\vec{x} \in {}^{\omega}S$ *satisfying uniqueness of products. Then there exists a nonprincipal strongly productive ultrafilter on* $S$*.*

*Proof.* It suffices to construct the ultrafilter on the subsemigroup $S'$ generated by the range of the sequence $\vec{x}$. Notice that $S'$ is countable, so we can enumerate all subsets of $S'$ in a $\mathfrak{c}$-sequence, $\langle A_\alpha \big| \alpha < \mathfrak{c}\rangle$. We recursively construct $\omega$-sequences $\vec{x}_\alpha$ that are product subsystems of $\mathrm{FP}(\vec{x})$ (note that the $\vec{x_\alpha}$ will automatically satisfy uniqueness of products) such that, for every $\alpha < \mathfrak{c}$:

(i) $(\exists B \in \{A_\alpha, S' \setminus A_\alpha\})(\mathrm{FP}(\vec{x}_\alpha) \subseteq B)$

(ii) $\{\mathrm{FP}_n(\vec{x_\xi}) \big| \xi < \alpha \wedge n < \omega\}$ has the strong finite intersection property.

We assume (by induction) that we have already constructed the $\vec{x_\xi}$, for $\xi < \alpha$, satisfying (i) and (ii); and show how to construct $\vec{x_\alpha}$. Let us further assume (innocuously) that $A_0 = S'$ and $\vec{x_0} = \vec{x}$. Corollary 1.7 together with clause (ii) ensures that there exists an idempotent ultrafilter $q * q = q \in S^*$ containing all of the $\mathrm{FP}_n(\vec{x_\xi})$ for $n < \omega$ and $\xi < \alpha$. We let $B \in \{A_\alpha, S' \setminus A_\alpha\}$ be such that $B \in q$. We will show how to construct (by using the hypothesis that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$) a sequence

$\vec{y} = \langle y_n \big| n < \omega\rangle$ of elements of $S'$ which is a product subsystem of $\vec{x}$ and such that $\mathrm{FP}(\vec{y}) \subseteq B$, and for every finitely many $\xi_1, \dots, \xi_k < \alpha$ and $n_1, \dots, n_k < \omega$, the sequence $\vec{y}$ contains infinitely many elements from $\bigcap\limits_{i=1}^{k} \mathrm{FP}_{n_i}(\vec{x_{\xi_i}})$. If we succeed to do this, it is clear that making $\vec{x_\alpha} = \vec{y}$ does the job and allows us to continue with the induction.

Hence we define the partial order $\mathbb{P}$ whose elements are all finite product subsystems of $\vec{x}$, $\langle y_i \big| i < k\rangle$, such that $\mathrm{FP}(\langle y_i \big| i < k\rangle) \subseteq B^\star$. The ordering is end-extension. Since $S'$ is countable (hence forcing equivalent to Cohen forcing) and we are assuming that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, then we have Martin's axiom satisfied for this particular forcing notion. We fix finitely many ordinals $\xi_1, \dots, \xi_k < \alpha$ and finite ordinals $n, n_1, \dots, n_k < \omega$ and claim that the set

$$D(\xi_1, \dots, \xi_k; n; n_1, \dots, n_k) = \left\{ \langle y_i | i < n\rangle \in \mathbb{P} \middle| \left| \{y_i | i < n\} \cap \left( \bigcap_{i=1}^{k} \mathrm{FP}_{n_i}(\vec{x_{\xi_i}}) \right) \right| \geq n \right\}$$

is dense in $\mathbb{P}$. If we prove the claim, we will be done, since there are only $|\alpha| < \mathfrak{c} = \mathrm{cov}(\mathcal{M})$ many such dense sets, so it is possible to find a filter on $\mathbb{P}$ meeting them all, and this filter will yield our desired sequence $\vec{y}$. So let us prove that $D(\xi_1, \dots, \xi_k; n; n_1, \dots, n_k)$ is dense in $\mathbb{P}$. Grab any condition $\langle y_i | i < k\rangle \in \mathbb{P}$. If $y_{k-1} = \prod_{j \in a} x_j$ then we let $m = \max(a) + 1$. Since our condition was assumed to be a product subsystem of $\vec{x}$, then every $y_i$ can be written as a product that only contains $x_j$ with $j < m$ (if $k = 0$ then we just let $m = 0$). Now note that the set

$$\mathrm{FP}_m(\vec{x}) \cap B^* \cap \left( \bigcap_{y \in \mathrm{FP}(\langle y_i | i < k \rangle)} y^{-1} * B^\star \right) \cap \left( \bigcap_{i=1}^{k} \mathrm{FP}_{n_i}(\vec{x_{\xi_i}}) \right) \in q.$$

Looking back at the proof of Theorem 2.2 and the paragraph that follows that theorem, we see that there exists an infinite sequence $\vec{z}$ which is a product subsystem of $\vec{x}$ and such that $\mathrm{FP}(\vec{z})$ is contained in the above set. Now we only need to grab the first $n$ elements $z_0, \ldots, z_{n-1}$ of the sequence $\vec{z}$ and extend our original condition to the condition $\vec{w} = \langle y_i | i < k \rangle \frown \langle z_j | j < n \rangle$. It is clear from the choice of $\vec{z}$ that $\mathrm{FP}(\vec{w}) \subseteq B^\star$, so $\vec{w} \in \mathbb{P}$ and actually $\vec{w} \in D(\xi_1, \ldots, \xi_k; n; n_1, \ldots, n_k)$ and we are done.

In the end, clearly the family $\{\mathrm{FP}_n(\vec{x_\alpha}) | \alpha < \mathfrak{c} \wedge n < \omega\}$ is a free ultrafilter base, generating a nonprincipal strongly productive ultrafilter $p \in S^*$. □

**Corollary 2.9.** *Let $S$ be a right cancellative and weakly left cancellative semigroup. Then, assuming* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*, there exist strongly productive ultrafilters on $S$.*

*Proof.* Proposition 2.7 and Theorem 2.8. □

The existence, under certain set-theoretic assumptions in addition to $\mathsf{ZFC}$, of strongly summable ultrafilters was first established by Hindman in the case of $\mathbb{N}$ [15, Theorem 3.3], on the Boolean group by Malyhin [30, Theorem 3], and finally on all abelian groups by Hindman, Protasov and Strauss [19, Theorem 2.8]. The reader might wonder whether it really is necessary to assume any additional hypothesis to the ZFC axioms. It was shown by Blass and Hindman [6, Theorem 2] that such a hypothesis is really necessary on $\mathbb{N}$, and later on Hindman, Protasov and Strauss [19, Theorem 3.6] extended that result to all abelian groups. In Chapter 4 we will show in a couple of different ways that, for a lot of semigroups (in particular for all abelian groups), the existence of strongly productive ultrafilters cannot be proved in ZFC. It is, however, still possible that we could eventually find some nontrivial (i.e. more complicated than those from Example 2.6) ZFC examples of semigroups that carry a strongly summable ultrafilter.

## 2.3 Superstrongly Productive Ultrafilters and IP-regular Sets

In this section we aim to prove that in most reasonable semigroups, all strongly productive ultrafilters are idempotent.

We will first introduce an apparent strengthening of the notion of a strongly productive ultrafilter, which will make the situation more comfortable with regards to talking about idempotents.

**Definition 2.10.** If $S$ is a semigroup, we say that a nonprincipal ultrafilter $p \in S^*$ is **superstrongly productive** (and if the semigroup is additively denoted then we say **superstrongly summable**) if for every $A \in p$ there exists an $\omega$-sequence $\vec{x}$ of elements of $S$ with the property that $\mathrm{FP}(\vec{x}) \subseteq A$ and $(\forall n < \omega)(\mathrm{FP}_n(\vec{x}) \in p)$.

The main reason for stating the preceding definition is that it allows us to easily prove the following proposition.

**Proposition 2.11.** *Let $S$ be a semigroup, and let $p \in S^*$ be a superstrongly productive ultrafilter. Then $p$ is idempotent.*

*Proof.* Given any fixed $A \in p$, pick an $\omega$-sequence $\vec{x_A}$ of elements of $S$ such that $\mathrm{FP}(\vec{x_A}) \subseteq A$ and $(\forall n < \omega)(\mathrm{FP}_n(\vec{x_A}) \in p)$. Then Corollary 1.7 ensures that the collection

$$\bigcap_{A \in p} \mathrm{FP}_\infty(\vec{x_A}) = \{q \in \beta S \big| \{\mathrm{FP}_n(\vec{x_A}) \big| A \in p \wedge n < \omega\} \subseteq q\}$$

is a subsemigroup of $S^*$, which means that it is closed under the semigroup operation $*$. However, it is readily checked that $\bigcap_{A \in p} \mathrm{FP}_\infty(\vec{x_A}) = \{p\}$, whereby we must have that $p * p = p$. □

In order to analyse strongly and superstrongly productive ultrafilters on semigroups, it is useful to introduce the following notions. This notions capture, in a sense, some combinatorial-algebraic idea of "largeness".

**Definition 2.12.** A subset $A$ of a semigroup $S$ is called an **IP-set** if it contains an FS-set, this is, if there exists an $\omega$-sequence $\vec{x}$ of elements of $S$ with infinite range such that $\mathrm{FP}(\vec{x}) \subseteq A$.

This notion allows us to reformulate Hindman's theorem in a much less verbose way: Hindman's theorem, in its most general version (this is, as in Theorem 2.2) simply states that, in any semigroup, the family of IP-sets is partition regular. Those ultrafilters all of whose elements are IP-sets are what we called weakly productive ultrafilters in the previous section.

We will now introduce a notion which will be crucial for establishing the idempotency of strongly productive ultrafilters.

**Definition 2.13.** Given a semigroup $S$, a subset $A \subseteq S$ is said to be **IP-regular** if, whenever we have an $\omega$-sequence $\vec{x}$ such that $\mathrm{FP}(\vec{x}) \subseteq A$, then the set

$$x_0 * \mathrm{FP}_1(\vec{x})$$

is not an IP-set.

Notice that singletons $\{x\}$ are (vacuously) IP-regular if and only if $x$ is not an idempotent. The language of IP-regular sets is at the very core of our proof of idempotency of strongly productive ultrafilters.

**Theorem 2.14.** *Let $S$ be a semigroup and $p \in S^*$ a strongly productive ultrafilter. If there is an IP-regular set $A \subseteq S$ such that $A \in p$, then $p$ is superstrongly productive.*

*Proof.* Let $S, A, p$ be as in the hypothesis, and let $B \in p$. Since $p$ is strongly productive, we can pick an $\omega$-sequence $\vec{x}$ of elements of $S$ such that $p \ni \mathrm{FP}(\vec{x}) \subseteq$

$A \cap B$. We will argue that $\mathrm{FP}_n(\vec{x}) \in p$ for every $n < \omega$. Recall that strongly summable ultrafilters, being in particular weakly summable, have the property that they consist of IP-sets only. Then notice that

$$\mathrm{FP}(\vec{x}) = \{x_0\} \cup (x_0 * \mathrm{FP}_1(\vec{x})) \cup (\mathrm{FP}_1(\vec{x})),$$

whereby, given that $p$ is nonprincipal (hence $\{x_0\} \notin p$) and that $A$ is IP-regular (hence $x_0 * \mathrm{FP}_1(\vec{x})$ is not an IP-set and so cannot be an element of $p$), it follows that $\mathrm{FP}_1(\vec{x}) \in p$. Repeated use of this argument allows us to conclude by induction on $n$ that $\mathrm{FP}_n(\vec{x}) \in p$ for every $n < \omega$. □

**Corollary 2.15.** *Let $S$ be a semigroup and $p \in S^*$ a strongly productive ultrafilter. If there is an IP-regular set $A \subseteq S$ such that $A \in p$, then $p$ is idempotent.*

*Proof.* Theorem 2.14 and Proposition 2.11. □

We now introduce the notation $E(S)$ for the set of idempotent elements of a semigroup $S$. This notation is fairly standard and will allow us to state the following corollary.

**Corollary 2.16.** *Let $S$ be a semigroup such that $E(S)$ is finite and $S \setminus E(S)$ can be partitioned into finitely many IP-regular cells. Then every strongly productive ultrafilter on $S$ is idempotent.*

*Proof.* The hypothesis implies that every strongly productive ultrafilter on $S$ is either in $E(S)$ or a nonprincipal ultrafilter that contains an IP-regular set, whereby Corollary 2.15 does the job. $\square$

If $G$ is a group with identity $e$, then $E(G) = \{e\}$ and so the previous corollary lets us conclude that, if $G \setminus \{e\}$ can be partitioned into finitely many IP-regular cells, then every strongly productive ultrafilter on $G$ is idempotent. We will show an argument that follows this line of reasoning in the next two sections.

## 2.4 Strongly Summable Ultrafilters on $\bigoplus_{\alpha<\kappa} \mathbb{T}$

Throughout this section, $G$ will always denote the group $\bigoplus_{\alpha<\kappa} \mathbb{T}$ (the definition of $\mathbb{T}$, as well as some notations for working with $G$, can be found in Section 1.1), for a given (infinite) cardinal $\kappa$. It is now time to "do the hard work" by proving that (if we denote the identity element of $G$ by 0) the set $G \setminus \{0\}$ can be partitioned into finitely many (at most 15, actually) IP-regular pieces. In the next section we extract a host of consequences of this fact, which by themselves will constitute an explanation of why the group $G$ is so important. The reader should bear in mind that, since $G$ is in fact abelian and additively denoted, we will be talking about strongly summable (rather than strongly productive) ultrafilters, and FS-sets (rather than FP-sets).

First of all, we introduce a notion that will be reasonably helpful when establishing that certain subsets of $G$ are IP-regular.

**Definition 2.17.** Given a subset $A \subseteq G$ and an ordinal $\alpha$, we say that a function $\rho : A \longrightarrow \alpha$ is a **rank function** if, whenever $\vec{x}$ is a sequence of elements of $G$ such that $\mathrm{FS}(\vec{x}) \subseteq A$, the following two conditions are satisfied:

(i) The restriction $\rho \upharpoonright \{x_n \big| n \in \omega\}$ of $\rho$ to the range of the sequence $\vec{x}$ is finite-to-one, and

(ii) if $\rho(x_n) \geq \rho(x_0)$ for every $n < \omega$ then $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set.

As an example, suppose that $\rho : A \longrightarrow \alpha$ (where $A \subseteq G$ and $\alpha$ is an ordinal) is a function such that $\rho(x + y) = \min\{\rho(x), \rho(y)\}$ for all $x, y \in A$; and furthermore $\rho(x) \neq \rho(y)$ whenever $x, y, x+y \in A$. Then it is easy to see that $\rho$ is a rank function on $A$. Later in this section we will encounter some functions $\rho$ that satisfy these exact properties. The relevance of rank functions for the proof of our main result is stated in the following lemma.

**Lemma 2.18.** *If we have a subset $A \subseteq G$ such that there is a rank function $\rho : A \longrightarrow \alpha$ on it ($\alpha$ an ordinal), then $A$ is IP-regular.*

*Proof.* Suppose that $\vec{x}$ is an $\omega$-sequence of elements of $G$ such that $\mathrm{FS}(\vec{x}) \subseteq A$. We claim that $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set. Since $\rho \upharpoonright \mathrm{ran}(\vec{x})$ is finite-to-one, it is possible to pick a permutation $\sigma$ of $\omega$ such that, whenever $n < m < \omega$,

$$\rho(x_{\sigma(n)}) \leq \rho(x_{\sigma(m)}).$$

Let the $\omega$-sequence $\vec{y}$ be defined by $y_n = x_{\sigma(n)}$ (hence if $n < m < \omega$ then $\rho(y_n) \leq \rho(y_m)$). Now observe that, if $k = \sigma^{-1}(0)$ (so that $x_0 = y_k$), then

$$x_0 + \mathrm{FS}_1(\vec{x}) = \Big(y_k + \mathrm{FS}_{k+1}(\vec{y})\Big) \cup \Big(y_k + \mathrm{FS}(\langle y_i \big| i < k\rangle)\Big) \cup \Big(y_k + \mathrm{FS}(\langle y_i \big| i < k\rangle) + \mathrm{FS}_{k+1}(\vec{y})\Big).$$

Since $\rho$ is a rank function and $\rho(y_n) \geq \rho(y_k)$ for $n \geq k$, it follows that

$$y_k + \mathrm{FS}_{k+1}(\vec{y})$$

is not an IP-set. Now for every $y = \sum_{n\in a} y_n \in \mathrm{FS}(\langle y_i \big| i < k\rangle)$, if $m = \min(a)$ then we have that

$$y_k + y + \mathrm{FS}_{k+1}(\vec{y}) \subseteq y_m + \mathrm{FS}_{m+1}(\vec{y})$$

and since the latter is not an IP-set (because $\rho$ is a rank function and $\rho(y_n) \geq \rho(y_m)$ for $n \geq m$), neither is the former. Hence the set $y_k + \mathrm{FS}(\langle y_i \big| i < k\rangle) + \mathrm{FS}_{k+1}(\vec{y})$, being a finite union of sets that are not IP, is itself not an IP-set (because of Hindman's theorem). Finally, the set $y_k + \mathrm{FS}(\langle y_i \big| i < k\rangle)$ is finite and hence not an IP-set. Using Hindman's theorem again, we conclude that

$$x_0 + \mathrm{FS}_1(\vec{x})$$

is not an IP-set, because it is a finite union of non-IP-sets, and we are done. $\square$

Recall that, in any abelian group, the only idempotent element is the identity of the group. Thus, as we remarked at the end of the previous section, in order to prove that every strongly summable ultrafilter on $G$ is superstrongly summable, it suffices to partition $G \setminus \{0\}$ into finitely many IP-regular pieces, because of Corollary 2.16. We aim to do this in the following series of lemmas.

**Lemma 2.19.** *Let*

$$B = \left\{ x \in G \setminus \{0\} \middle| (\forall \alpha < \kappa) \left( \pi_\alpha(x) \in \left\{ 0, \frac{1}{2} \right\} \right) \right\}.$$

*Then $B$ is IP-regular.*

*Proof.* Observe that the subgroup $B \cup \{0\}$ of $G$ really is (isomorphic to) $\mathbb{B}_\kappa$, the Boolean group on cardinality $\kappa$. Hence, as was explained in Section 1.4, this subgroup has a structure of $\kappa$-dimensional vector space over the field with two elements $\mathbb{F} = \mathbb{Z}/\mathbb{Z}_2$, and if $\vec{x}$ is a sequence in $B$ then $\mathrm{FS}(\vec{x}) \cup \{0\}$ is the vector space generated by $\vec{x}$. Moreover the sequence $\vec{x}$ is linearly independent if and only if $0 \notin \mathrm{FS}(\vec{x})$. Thus if $\vec{x}$ is a sequence in $B$ such that $\mathrm{FS}(\vec{x}) \subseteq B$ then $\vec{x}$ is a linearly independent sequence, and hence every element of $\mathrm{FS}(\vec{x})$ can be written in a unique way as a sum of elements of the sequence $\vec{x}$. In particular $x_0 + \mathrm{FS}_1(\vec{x})$ consists of those finite sums $\sum_{i \in a} x_i$ such that $0 \in a$. Given two finite $a, b \subseteq \omega$ such that $0 \in a \cap b$, we get that (since under these conditions $0 \notin a \bigtriangleup b$)

$$\sum_{i \in a} x_i + \sum_{i \in b} x_i = \sum_{i \in a \bigtriangleup b} x_i \notin x_0 + \mathrm{FS}_1(\vec{x}).$$

This implies that $x_0 + \mathrm{FS}_1(\vec{x})$ cannot be an IP-set, hence $B$ is IP-regular. $\square$

It remains to show now that $C = G \setminus (B \cup \{0\})$ (where $B$ is defined as in the previous lemma) can be partitioned into finitely many IP-regular cells. Elements $x \in C$ have order strictly greater than 2, thus there is at least one $\alpha < \kappa$ such that $\pi_\alpha(x) \notin \left\{0, \frac{1}{2}\right\}$. Therefore it is possible to define the function $\mu : C \to \kappa$ by

$$\mu(x) = \min \left\{ \alpha < \kappa \middle| \pi_\alpha(x) \notin \left\{0, \frac{1}{2}\right\} \right\}.$$

Consider the partition of $C$ into four cells $C = C_1 \cup C_2 \cup C_3$, where

$$C_1 = \left\{ x \in C \middle| \pi_{\mu(x)}(x) = \frac{1}{4} \right\},$$
$$C_3 = \left\{ x \in C \middle| \pi_{\mu(x)}(x) = \frac{3}{4} \right\}, \text{ and}$$
$$C_2 = \left\{ x \in C \middle| \pi_{\mu(x)}(x) \notin \left\{ \frac{1}{4}, \frac{3}{4} \right\} \right\}.$$

**Lemma 2.20.** *The functions $\mu \restriction C_1$ and $\mu \restriction C_3$ are rank functions (on $C_1$ and $C_3$, respectively).*

*Proof.* We first prove the statement for $\mu \restriction C_1$. So suppose that $\vec{x}$ is a sequence in $G$ such that $\mathrm{FS}(\vec{x}) \subseteq C_1$. We will show that the function $\mu \restriction \{x_n | n < \omega\}$ is at-most-two-to-one, in particular finite-to-one. This is because if $n, m, k < \omega$ are three distinct numbers such that $\mu(x_n) = \mu(x_m) = \mu(x_k) = \alpha$, then for $\beta < \alpha$ we get that $\pi_\beta(x_n + x_m + x_k) \in \left\{0, \frac{1}{2}\right\}$ (because so are $\pi_\beta(x_n), \pi_\beta(x_m), \pi_\beta(x_k)$); while on the other hand $\pi_\alpha(x_n+x_m+x_k) = \frac{3}{4} \notin \left\{0, \frac{1}{2}\right\}$. This shows that $\mu(x_n+x_m+x_k) = \alpha$ and $x_n + x_m + x_k \in C_3$, which is a contradiction.

Now add in the assumption that $\alpha = \mu(x_0) \leq \mu(x_k)$ for every $k < \omega$. By the previous paragraph, there is at most one $n \in \mathbb{N}$ such that $\mu(x_n) = \mu(x_0) = \alpha$. We thus split the proof into two cases, according to whether or not there exists such an $n$. In the case where this $n$ exists, the first thing to notice is that for each $k \in \omega \setminus \{0, n\}$, $\pi_\alpha(x_k) = 0$. This is because otherwise, since $\mu(x_k) > \alpha$ we would have that $\pi_\alpha(x_k) = \frac{1}{2}$ and so $\pi_\alpha(x_0 + x_k) = \frac{3}{4}$; so arguing as in the previous paragraph we get that $x_0 + x_k \in C_3$, a contradiction. Now write

$$x_0 + \mathrm{FS}_1(\vec{x}) = \{x_0 + x_n\} \cup \left(x_0 + \mathrm{FS}(\langle x_k \big| k \in \omega \setminus \{0, n\}\rangle)\right)$$
$$\cup \left(x_0 + x_n + \mathrm{FS}(\langle x_k \big| k \in \omega \setminus \{0, n\}\rangle)\right).$$

Clearly $\{x_0 + x_1\}$ is not an IP-set, as it is finite. Now since $\pi_\alpha(x_k) = 0$ for $k \notin \omega \setminus \{0, n\}$, it follows that every element $x \in x_0 + \mathrm{FS}(\langle x_k \big| k \in \omega \setminus \{0, n\}\rangle)$ must satisfy $\pi_\alpha(x) = \frac{1}{4}$, which implies that $x_0 + \mathrm{FS}(\langle x_k \big| k \in \omega \setminus \{0, n\}\rangle)$ cannot contain the sum of any two of its elements and consequently it is not an IP-set. By the same token, every element $x \in x_0 + x_n + \mathrm{FS}(\langle x_k \big| k \in \omega \setminus \{0, n\}\rangle)$ satisfies that $\pi_\alpha(x) = \frac{1}{2}$, so this set is (by the same argument) not an IP-set either. Hence $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set.

Now if there is no such $n$, i.e. if $\mu(x_k) > \mu(x_0) = \alpha$ for all $k > 0$, then it is possible to repeat the argument from the previous paragraph except that we delete every reference to $x_n$. This is, we first get that $\pi_\alpha(x_k) = 0$ for all $k > 0$; from which we derive that every element $x \in x_0 + \mathrm{FS}_1(\vec{x})$ satisfies $\pi_\alpha(x) = \frac{1}{4}$. This implies that the set $x_0 + \mathrm{FS}_1(\vec{x})$ cannot be an IP-set, hence in any case $\mu$ is a rank function on $C_1$.

In order to see that the same holds for $\mu \restriction C_3$, one just needs to consider the fact that the function $t \longmapsto -t$ is an automorphism of $G$ which maps $C_1$ onto $C_3$ and preserves $\mu$. $\square$

Given the previous results, in order to have our desired result it suffices to show that $C_2$ is a union of finitely many IP-regular sets.

Define

$$Q_{i,j} = \left\{ x \in C_2 : \pi_{\mu(x)}(x) \in \bigcup_{m\in\omega} \left[ \frac{i}{4} + \frac{1}{2^{3m+j+3}}, \frac{i}{4} + \frac{1}{2^{3m+j+2}} \right) \right\}$$

for $i \in 4$ and $j \in 3$. This defines a partition $C_2 = \bigcup_{i\in 4} \bigcup_{j\in 3} Q_{i,j}$ of $C_2$ into 12 cells. We claim that for every $i \in 4$ and $j \in 3$, the set $Q_{i,j}$ is IP-regular. This will follow from the following lemma.

**Lemma 2.21.** *Consider $\kappa \times \omega$ equipped with the lexicographic order (which well-orders this set). The function $\rho : Q_{i,j} \to \kappa \times \omega$ defined by $\rho(x) = (\mu(x), m)$ where $m$ is the unique element of $\omega$ such that*

$$\pi_{\mu(x)}(x) \in \left[ \frac{i}{4} + \frac{1}{2^{3m+j+3}}, \frac{i}{4} + \frac{1}{2^{3m+j+2}} \right)$$

*is a rank function on* $Q_{i,j}$.

*Proof.* In order to simplify notation, we will run the proof in the case when $i = j = 0$ and mention in passing that the proof in the other cases is entirely analogous. We mentioned earlier (in the example right after Definition 2.17) that it suffices to show the following: if $x$ and $y$ are such that $x, y, x + y \in Q_{0,0}$, then $\rho(x) \neq \rho(y)$ and $\rho(x + y) = \min\{\rho(x), \rho(y)\}$. So suppose that $x, y \in Q_{0,0}$ are such that $x + y \in Q_{0,0}$ and assume by contradiction that $\rho(x) = \rho(y) = (\alpha, m)$. This entails that $\mu(x + y) = \alpha$ and

$$\pi_\alpha(x), \pi_\alpha(y) \in \left[\frac{1}{2^{3m+3}}, \frac{1}{2^{3m+2}}\right)$$

and hence

$$\pi_\alpha(x + y) \in \left[\frac{1}{2^{3m+2}}, \frac{1}{2^{3m+1}}\right).$$

If $m = 0$ then

$$\pi_\alpha\left(x + y\right) \in \left[\frac{1}{4}, \frac{1}{2}\right),$$

which implies that

$$x + y \in C_1 \cup Q_{1,0} \cup Q_{1,1} \cup Q_{1,3}$$

On the other hand, if $m > 0$ then

$$\pi_\alpha(x + y) \in \left[\frac{1}{2^{3(m-1)+2+3}}, \frac{1}{2^{3(m-1)+2+2}}\right)$$

and therefore

$$x + y \in Q_{0,2}.$$

In either case one obtains a contradiction from the assumption that $x + y \in Q_{0,0}$. This concludes the proof that $\rho(x) \neq \rho(y)$. We now claim that $\rho(x + y) = \min\{\rho(x), \rho(y)\}$. Define $\rho(x) = (\alpha, m)$ and $\rho(y) = (\beta, n)$. Let us first consider the case when $\alpha = \beta$ and without loss of generality $m > n$. In this case

$$\pi_\xi(x + y) \in \left\{0, \frac{1}{2}\right\}$$

for $\xi < \alpha$, while

$$\pi_\alpha(x + y) \in \left[\frac{1}{2^{3m+3}} + \frac{1}{2^{3n+3}}, \frac{1}{2^{3m+2}} + \frac{1}{2^{3n+2}}\right)$$

where

$$\frac{1}{2^{3m+2}} + \frac{1}{2^{3n+2}} < \frac{1}{2^{3n+1}} < \frac{1}{2^{3(n-1)+3}}.$$

This shows that $\rho(x + y) = (\alpha, n) = \min\{\rho(x), \rho(y)\}$. Let us now consider the case when $\alpha \neq \beta$ and without loss of generality $\alpha > \beta$. In this case

$$\pi_\xi(x + y) \in \left\{0, \frac{1}{2}\right\}$$

for $\xi < \beta$ while

$$\pi_\beta(x) = 0$$

(because if not then $\pi_\beta(x) = \frac{1}{2}$ and that would imply that $x + y \in \bigcup_{j \in 3} Q_{1,j}$), and hence

$$\pi_\beta(x + y) = \pi_\beta(y).$$

This shows that $\rho(x+y) = (\beta, n) = \min\{\rho(x), \rho(y)\}$ and hence concludes the proof of the fact that $\rho$ is a rank function on $Q_{0,0}$. □

**Corollary 2.22.** *The group $G$ can be partitioned, after removing the identity element, into at most* 15 *IP-regular cells.*

*Proof.* Put together Lemmas 2.19, 2.20 and 2.21. □

We will now proceed to extract the consequences of Corollary 2.22 in the following section.

## 2.5 Consequences on Miscellaneous Semigroups

Recall that $E(S)$ denotes the set of idempotent elements of a given semigroup $S$. Throughout this section, we will let $\Gamma$ stand for the class of all semigroups $S$ such that $E(S)$ is finite and $S \setminus E(S)$ can be partitioned into finitely many IP-regular cells. The importance of the class $\Gamma$ is, of course, that elements $S \in \Gamma$ satisfy that every strongly productive ultrafilter $p \in S^*$ is idempotent, but the reason that we define $\Gamma$ as we do is that, as we will see in this section, the class $\Gamma$ has some significant closure properties. The main result from the previous section (namely, Corollary 2.22) establishes that, for every infinite cardinal $\kappa$, the group $\bigoplus_{\alpha<\kappa} \mathbb{T} \in \Gamma$.

Suppose that $S, T$ are semigroups and $f : S \to T$ is a semigroup homomorphism. It is easy to see that, if $A \subseteq T$ is IP-regular, then so is $f^{-1}[A]$. This implies that, if

$T \setminus E(T)$ can be partitioned into finitely many IP-regular sets, then so can $S \setminus E(S)$. More concretely, we have the following straightforward lemma.

**Lemma 2.23.** *Let $S$ and $T$ be semigroups such that $E(S)$ is finite and $T \in \Gamma$. Let $f$ be a semigroup homomorphism such that $f^{-1}[E(T)]$ is a subsemigroup and moreover $f^{-1}[E(T)] \in \Gamma$. Then $S \in \Gamma$ as well.*

Hindman, Protasov and Strauss [19, Theorem 2.3] proved that every strongly summable ultrafilter on an abelian group is idempotent. The proof was significantly involved, and an attempt to simplify that proof was what originally triggered the study [13]. The next corollary closes the circle by establishing that our method embraces the result of Hindman, Protasov and Strauss as a particular case.

**Corollary 2.24.** *Every commutative cancellative semigroup belongs to the class $\Gamma$. In particular every abelian group belongs to $\Gamma$.*

*Proof.* It is well-known that every commutative cancellative semigroup can be embedded in an abelian group, which in turn (as mentioned in the Introduction) can be embedded into the group $\bigoplus_{\alpha<\kappa} \mathbb{T}$ for some infinite cardinal $\kappa$. Hence Corollary 2.22 together with Lemma 2.23 do the job together. □

In the particular context of groups, Lemma 2.23 guarantees that the extension of a group $G \in \Gamma$ by another group $H \in \Gamma$ is itself an element of $\Gamma$. In other words, if $K$ is a group containing a normal subgroup $H \in \Gamma$ in such a way that $K/H \in \Gamma$, then

$K \in \Gamma$ as well (by Lemma 2.23 applied to the projection mapping $\pi : K \longrightarrow K/H$). Thus, for example, one can consider groups admitting a subnormal series with factor groups that are elements of $\Gamma$. Recall that a subnormal series of a group $G$ is a finite sequence $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \langle e \rangle$ of subgroups of $G$ such that for each $k < n$, $G_{k+1} \trianglelefteq G_k$. The quotients $G_k/G_{k+1}$ are the **factor groups** of the series.

**Proposition 2.25.** *Let $G$ be a group admitting a subnormal series $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \langle e \rangle$ such that all of the factor groups $G_k/G_{k+1}$ are elements of $\Gamma$. Then $G \in \Gamma$.*

*Proof.* Lemma 2.23 (used as in the remark from the previous paragraph) and induction on the length of the subnormal series. □

**Corollary 2.26.** *All solvable groups are elements of $\Gamma$.*

*Proof.* Recall that solvable groups are exactly those groups that admit a subnormal series in which all factor groups are abelian. Hence, Corollary 2.24 and Proposition 2.25 together do the job. □

The class $\Gamma$ is, in fact, reasonably large. Notice that it has nice closure properties: it is closed with respect to taking subsemigroups (Lemma 2.23 applied to the inclusion mapping) and contains all finite semigroups. This, together with Proposition 2.25, means that all virtually solvable groups and their subgroups belong to the class $\Gamma$. We will see next that $\Gamma$ is (in a sense) closed with respect to free products.

**Proposition 2.27.** *Let $S, T$ be semigroups. If $S, T \in \Gamma$ and $T$ has no idempotent elements, then the free product $S * T \in \Gamma$.*

*Proof.* Denote by $T_e$ the semigroup obtained from $T$ by adjoining an identity element $e$. Consider the semigroup homomorphism $f : S * T \longrightarrow T_e$ given by sending every word $w$ to the product of the sequence of letters appearing in $w$ that are elements of $T$ (in the same order in which they appear in $w$, and remembering that the empty product is just the identity element $e$). Observe that $f^{-1}[\{e\}]$ is isomorphic to $S \in \Gamma$. Hence $S * T \in \Gamma$ by Lemma 2.23. □

The particular case of Proposition 2.27 when $S = T = \mathbb{N}$ yields that the free semigroup on 2 generators is an element of $\Gamma$. In fact, we can obtain a much more general result.

**Theorem 2.28.** *Every free semigroup $S$ (regardless of the number of generators) is an element of $\Gamma$.*

*Proof.* Consider the function $\ell : S \longrightarrow \mathbb{N}$ mapping each word $w \in S$ to its length. Since $\ell$ is a semigroup homomorphism and $\mathbb{N} \in \Gamma$ (being a cancellative abelian semigroup), the result follows from Lemma 2.23. □

Theorem 2.28 was originally proved by Hindman and Jones [18, Lemma 2.2] (in their hypothesis that $p$ is a *very* strongly productive ultrafilter, the "very" part is not really used). To the author, it was immediately apparent that the idea for the proof of Hindman and Jones could easily be applied to the more general context of the class $\Gamma$, as we just did in Theorem 2.28.

The intention here was just to show some nice examples of semigroups and groups that belong to the class $\Gamma$. A much broader class of semigroups has been proven to belong to $\Gamma$ in [13]. To the best of my knowledge, it is currently not known whether the existence of a semigroup $S$ and a nonprincipal strongly productive ultrafilter on $S$ that is not idempotent is consistent with the usual axioms of set theory. We believe that the answer should be negative, as was stated in the following conjecture from [13].

**Conjecture 2.29.** *If $S$ is any semigroup and $p \in S^*$ is strongly productive, then $p$ is idempotent.*

# 3 Sparseness and the Trivial Sums Property

In this chapter we prove that all strongly summable ultrafilters on abelian groups, and a restricted class of strongly productive ultrafilters on the free semigroup, are sparse and moreover they can only be decomposed as a sum (product in the case of the free semigroup) in a trivial way. The new results (i.e., the original contributions from the author) in this chapter can be found in [11] if they refer to abelian groups, and in [13] if they refer to the free semigroup.

## 3.1 Sparseness, Trivial Sums and Trivial Products

We will introduce an important notion. To motivate it, we urge the reader to think of the uniqueness of sums of a sequence $\vec{x}$ as a requirement that deals with combinations of elements of $\vec{x}$ with coefficients equal to 1.

**Definition 3.1.** Given an $n \in \mathbb{N}$, a sequence $\vec{x}$ on an abelian group $G$ is said to satisfy the **$n$-uniqueness of sums** if whenever $a, b \in [\omega]^{<\omega}$ and $\varepsilon : a \longrightarrow$

$\{1,2,\ldots,n\}, \delta : b \longrightarrow \{1,2,\ldots,n\}$ are such that

$$\sum_{i\in a} \varepsilon(i) x_i = \sum_{i\in b} \delta(i) x_i,$$

it must be the case that $a = b$ and $\varepsilon = \delta$.

The reader that is familiar with the notations used in Gowers's theorem will notice that $\vec{x}$ satisfies the $n$-uniqueness of sums if and only if the mapping

$$\varphi : \mathrm{FIN}_{[1,n]} \longrightarrow G$$
$$f \longmapsto \sum_{i<\omega} f(i) x_i$$

is injective. Notice that, in particular, if $\vec{x}$ satisfies $n$-uniqueness of sums then no element of $\vec{x}$ can have order $n$. Thus, in particular, Boolean groups do not contain sequences satisfying 2-uniqueness of sums. For the results of this thesis we only need to consider the case $n = 2$.

**Proposition 3.2.** *For a sequence $\vec{x}$ on an abelian group $G$, the following are equivalent.*

*(i) $\vec{x}$ satisfies the 2-uniqueness of sums.*

*(ii) Whenever $a, b, c, d \in [\omega]^{<\omega}$ are such that $a \cap b = \varnothing = c \cap d$, if*

$$2\sum_{n\in a} x_n + \sum_{n\in b} x_n = 2\sum_{n\in c} x_n + \sum_{n\in d} x_n$$

*then $a = c$ and $b = d$.*

*(iii) Whenever $a, b, c, d \in [\omega]^{<\omega}$ are such that*

$$\sum_{n\in a} x_n + \sum_{n\in b} x_n = \sum_{n\in c} x_n + \sum_{n\in d} x_n,$$

*it must be the case that $a \triangle b = c \triangle d$ and $a \cap b = c \cap d$.*

*Proof.* Straightforward. □

We now see an important application of this concept. The following result can be traced back to [6, Theorem 1] (that is, the cited result uses the same main ideas) although it appeared in full form as half of [20, Theorem 3.2].

**Theorem 3.3.** *Let $p$ be a strongly summable ultrafilter such that for some $\vec{x}$ satisfying 2-uniqueness of sums, $\mathrm{FS}(\vec{x}) \in p$. Then $p$ is additively isomorphic to a union ultrafilter.*

*Proof.* We just need to check that the mapping $\varphi$ given by $\varphi(\sum_{n\in a} x_n) = a$ sends $p$ to a union ultrafilter. So let $A \in q = \varphi(p)$. Pick a sequence $\vec{y}$ such that $p \ni \mathrm{FS}(\vec{y}) \subseteq \varphi^{-1}[A]$. Then $\varphi[\mathrm{FS}(\vec{y})] \subseteq A$. Now $\varphi^{-1}[A] \subseteq \mathrm{FS}(\vec{x})$, thus for each $n < \omega$ we can define $c_n \in [\omega]^{<\omega}$ by $c_n = \varphi(y_n)$ or, equivalently, by $y_n = \sum_{i\in c_n} x_i$. We claim that the family $C = \{c_n \big| n < \omega\}$ is pairwise disjoint. This is because if $n \neq m$, since $y_n + y_m \in \mathrm{FS}(\vec{y}) \subseteq \mathrm{FS}(\vec{x})$, then there must be a $c \in [\omega]^{<\omega}$ such that

$$\sum_{i\in c} x_i = y_n + y_m = \sum_{i\in c_n} x_i + \sum_{i\in c_m} x_i.$$

Since $\vec{x}$ satisfies 2-uniqueness of sums, by Proposition 3.2 we can conclude that $c = c_n \cup c_m$ and $c_n \cap c_m = \varnothing$. This argument shows at once that $C$ is a pairwise disjoint family, and that $\varphi(y_n + y_m) = c_n \cup c_m = \varphi(y_n) \cup \varphi(y_m)$. From this it is easy to prove by induction that $\varphi\left(\sum_{n\in a} y_n\right) = \bigcup_{n\in a} \varphi(y_n)$, for all $a \in [\omega]^{<\omega}$, hence $\varphi[\mathrm{FS}(\vec{y})] = \mathrm{F}\triangle(C)$, therefore $q \ni \mathrm{F}\triangle(C) \subseteq A$ and we are done. $\square$

We next present two old results to illustrate that sequences satisfying 2-uniqueness of sums are quite ubiquitous. These results will refer to the restricted context of ultrafilters on $\mathbb{N}$. Some of the relevant ideas for these results will be used later on, in Section 3.3. The first result is the following theorem, originally due to Blass and Hindman [6, Lemma 1C] (although at the time, the terminology of 2-uniqueness of sums was not in use).

**Theorem 3.4.** *Let $p \in \mathbb{N}^*$ be a strongly summable ultrafilter. Then, there exists a sequence $\vec{x}$ satisfying 2-uniqueness of sums such that* $\mathrm{FS}(\vec{x}) \in p$.

*Proof.* Partition $\mathbb{N}$ into the three cells $A_0, A_1, A_2$, where

$$A_i = \{n \in \mathbb{N} \big| \lfloor \log_2(n) \rfloor \equiv i \mod 3\}.$$

In order to properly visualize things, just notice that $A_i$ is the set of natural numbers whose binary expansion has its leftmost nonzero digit in a position (counting from the right) which is $i$ modulo 3. Pick an $i \in 3$ such that $A_i \in p$ and let $\vec{x}$ be such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A_i$. Now notice that for $j < k < \omega$, it is not possible that

$\lfloor \log_2(x_j) \rfloor = \lfloor \log_2(x_k) \rfloor$ (otherwise we would get $\lfloor \log_2(x_j + x_k) \rfloor = \lfloor \log_2(x_j) \rfloor + 1$ thus $x_j + x_k \in A_{i+1 \mod 3}$, a contradiction). Hence, by reordering the sequence if necessary, we may assume that for $j < k < \omega$ we have $\lfloor \log_2(x_j) \rfloor < \lfloor \log_2(x_k) \rfloor$. This implies that, for all $n < \omega$, $x_{n+1} > 4x_n$ (for since $\lfloor \log_2(x_n) \rfloor < \lfloor \log_2(x_{n+1}) \rfloor$ and $x_n, x_{n+1} \in A_i$, we actually get that $\lfloor \log_2(x_n) \rfloor < \lfloor \log_2(x_{n+1}) \rfloor + 2$), and the latter inequality allows us to prove by induction that, for all $n < \omega$, $x_{n+1} > 2\sum_{k \leq n} x_k$. This can easily be seen to imply that the sequence $\vec{x}$ satisfies 2-uniqueness of sums, and we are done. □

This allows us to conclude the following corollary which is [6, Theorem 1].

**Corollary 3.5.** *Every strongly summable ultrafilter on* $\mathbb{N}$ *is additively isomorphic to a union ultrafilter.* □

We will now see how the notion of 2-uniqueness of sums yields what is known in the literature as *strong maximal idempotent* elements in Čech-Stone compactification. The following result is [19, Theorem 4.3].

**Theorem 3.6.** *Let* $G$ *be an abelian group, and let* $p \in G^*$ *be a strongly summable ultrafilter such that* $\mathrm{FS}(\vec{x}) \in p$ *for some* $\vec{x}$ *satisfying 2-uniqueness of sums. If* $q \in G^*$ *is such that* $q + p = p$*, then it must be the case that* $q = p$.

*Proof.* Let $G, p, q, \vec{x}$ be as in the hypotheses and suppose that $q \neq p$. We will show that $q + p \neq p$. Start by choosing an $A \in p \setminus q$. Grab a sequence $\vec{y}$ such that

$p \ni \mathrm{FS}(\vec{y}) \subseteq A \cap \mathrm{FS}(\vec{x})$ (notice that $\vec{y}$ will automatically satisfy 2-uniqueness of sums). Then of course we have that $\mathrm{FS}(\vec{y}) \notin q$. We claim that

$$\{y \in G \big| -y + \mathrm{FS}(\vec{y}) \in p\} \subseteq \mathrm{FS}(\vec{y}),$$

so the set on the left of this expression cannot belong to $q$, but this means that $\mathrm{FS}(\vec{y})$ cannot belong to $q+p$ and therefore we must have $q+p \neq p$. In order to prove the claim, let $y \in G$ be such that $-y + \mathrm{FS}(\vec{y}) \in p$. Now $p$ is weakly summable, so the set $(-y+\mathrm{FS}(\vec{y})) \cap \mathrm{FS}(\vec{y}) \in p$ should be an IP-set and in particular it is possible to find two elements $z, w \in \mathrm{FS}(\vec{y})$ such that $y+z, y+w, z+w \in \mathrm{FS}(\vec{y})$. This means that we can pick elements $a, b, c, d \in [\omega]^{<\omega}$ such that $z = \sum_{n\in a} y_n$, $w = \sum_{n\in b} y_n$, $y + z = \sum_{n\in c} y_n$ and $y + w = \sum_{n\in d} y_n$. We thus get that

$$y = \sum_{n\in c} y_n - \sum_{n\in a} y_n = \sum_{n\in d} y_n - \sum_{n\in b} y_n,$$

which leads to

$$\sum_{n\in a} y_n + \sum_{n\in d} y_n = \sum_{n\in b} y_n + \sum_{n\in c} y_n.$$

Since $\vec{y}$ satisfies 2-uniqueness of sums, we conclude that $a \cap d = b \cap c$ and $a \triangle d = b \triangle c$. Now notice that, since $z + w \in \mathrm{FS}(\vec{y})$ and $\vec{y}$ satisfies 2-uniqueness of sums, we can conclude that $a \cap b = \varnothing$, which together with the previous equalities of sets leads to the conclusion that $a \subseteq c$. This implies that $y = \sum_{n\in c\setminus a} y_n \in \mathrm{FS}(\vec{y})$. □

The last three results are illustrations of the importance of the concept of 2-uniqueness of sums. We will now introduce a couple of further notions that are of central importance in this dissertation. Without fear of exaggerating, we can confidently state that the following two definitions are the most important ones within this thesis (at the very least, they are certainly the ones whose study originally prompted all of the results presented here).

**Definition 3.7.** Let $G$ be an abelian group.

(i) We say that an idempotent $p+p=p\in G^*$ has the **trivial sums property** if, whenever $q,r\in G^*$ are such that $q+r=p$, it must be the case that $q,r\in G+p$ (more specifically, there must exist an $x\in G$ such that $q=p+x$ and $r=-x+p$).

(ii) We say that a strongly summable ultrafilter $p\in G^*$ is **sparse** if for every $A\in p$ there exists a sequence $\vec{x}$ such that $\mathrm{FS}(\vec{x})\subseteq A$ and a subsequence $\vec{y}$ of $\vec{x}$ such that $\{x_n \big| n<\omega\}\setminus\{y_n\big| n<\omega\}$ is infinite and $\mathrm{FS}(\vec{y})\in p$. (Slang: "we can drop infinitely many generators of $\mathrm{FS}(\vec{x})$ and still remain within the ultrafilter").

These two notions are related by a result of Hindman, Protasov and Strauss [19, Theorem 4.5] stating that if $G\subseteq\mathbb{T}$ and $p\in G^*$ is sparse, then $p$ has the trivial sums property. A good portion of the remainder of this section will be devoted to presenting our own proof of the following result of Hindman, Steprāns and Strauss [20, Theorem 4.8] which relates the two notions in question with the 2-uniqueness of sums.

**Theorem 3.8.** *Let $G$ be an abelian group and let $p \in G^*$ be a sparse ultrafilter such that $\mathrm{FS}(\vec{x}) \in p$ for some sequence $\vec{x}$ satisfying the 2-uniqueness of finite sums. Then $p$ has the trivial sums property.*

We will now start working towards a proof of Theorem 3.8, and in order to do this we will need to introduce yet some more notation (which we will be able to completely forget once we are done with this proof). Given a sequence $\vec{x}$ of elements of our abelian group $G$, we denote by $\mathrm{FS}^{\pm}(\vec{x}) = \mathrm{FS}(\vec{x}) \cup (-\mathrm{FS}(\vec{x})) \cup (\mathrm{FS}(\vec{x}) - \mathrm{FS}(\vec{x}))$. In other words, $\mathrm{FS}^{\pm}(\vec{x})$ is the set consisting of all elements of the form $\sum_{n\in a} \varepsilon(n) x_n$, where $a \in [\omega]^{<\omega}$ and $\varepsilon : a \longrightarrow \{-1, 1\}$. Equivalently, $\mathrm{FS}^{\pm}(\vec{x})$ is the set of all elements of the form

$$\sum_{n\in a} x_n - \sum_{n\in b} x_n,$$

where $a, b \in [\omega]^{<\omega}$ and we can always assume without loss of generality that $a \cap b = \varnothing$. The following lemma establishes that the 2-uniqueness of sums is also, in a sense, some sort of "$\pm$-uniqueness of sums".

**Lemma 3.9.** *Let $\vec{x}$ be a sequence of elements of $G$ satisfying the 2-uniqueness of sums. Then, the representation of an element of $\mathrm{FS}^{\pm}(\vec{x})$ as a difference $\sum_{n\in a} x_n - \sum_{n\in b} x_n$ for two disjoint $a, b \in [\omega]^{<\omega}$ is unique.*

*Proof.* Suppose that $a, b, c, d \in [\omega]^{<\omega}$ are such that $a \cap b = \varnothing = c \cap d$ and

$$\sum_{n \in a} x_n - \sum_{n \in b} x_n = \sum_{n \in c} x_n - \sum_{n \in d} x_n.$$

We let $e = (a \cap c) \cup (b \cap d)$. We define $a' = a \setminus e$ and $c' = c \setminus e$, and similarly we let $b' = b \setminus e$ and $d' = d \setminus e$. Then we have that $a' \cap c' = \varnothing = b' \cap d'$. From the hypothesis, by cancelling all terms of the form $\pm x_n$, for $n \in e$, from both sides of the equation, we arrive at

$$\sum_{n \in a'} x_n - \sum_{n \in b'} x_n = \sum_{n \in c'} x_n - \sum_{n \in d'} x_n,$$

which implies that

$$\sum_{n \in a'} x_n + \sum_{n \in d'} x_n = \sum_{n \in c'} x_n + \sum_{n \in b'} x_n.$$

Now $\vec{x}$ has the 2-uniqueness of sums, so we can conclude from the previous equation that $a' \triangle d' = b' \triangle c'$ and $a' \cap d' = b' \cap c'$. But $a'$ is disjoint from both $b'$ and $c'$, and also $d'$ is disjoint from both $b'$ and $c'$; so the conclusion is that $a' \triangle d' = b' \triangle c' = a' \cap d' = b' \cap c' = \varnothing$. This certainly implies that $a' = b' = c' = d' = \varnothing$, which in turn yields that $a = c$ and $b = d$, and we are done. $\square$

At some point during the proof that we are currently working towards, we will also need the following property that sequences with the 2-uniqueness of sums satisfy.

**Lemma 3.10.** *Let $\vec{x}$ be a sequence of elements of $G$ satisfying the 2-uniqueness of sums. Let $a, b, c \in [\omega]^{<\omega}$ be pairwise disjoint such that*

$$2\sum_{n\in a} x_n + \sum_{n\in b} x_n - \sum_{n\in c} x_n \in \mathrm{FS}(\vec{x}).$$

*Then it must be the case that $a = c = \varnothing$.*

*Proof.* Let $d \in [\omega]^{<\omega} \setminus \varnothing$ be such that

$$2\sum_{n\in a} x_n + \sum_{n\in b} x_n - \sum_{n\in c} x_n = \sum_{n\in d} x_n.$$

Cancelling terms in common from both sides of this equation yields

$$2\sum_{n\in a'} x_n + \sum_{n\in b'} x_n - \sum_{n\in c} x_n = \sum_{n\in d'} x_n,$$

where $a' = a \setminus d$, $b' = (b \setminus d) \cup (a \cap d)$ and $d' = d \setminus (a \cup b)$. Note that $a', b', d'$ are pairwise disjoint. The last equation can be turned into

$$2\sum_{n\in a'} x_n + \sum_{n\in b'} x_n = \sum_{n\in d'} x_n + \sum_{n\in c} x_n,$$

whereby we can conclude, since $\vec{x}$ satisfies the 2-uniqueness of sums, that $a' = c \cap d'$ and $b' = c \bigtriangleup d'$. Since $a'$ is disjoint from both $c$ and $d'$, we conclude that $a' = \varnothing = c \cap d'$, so $c \bigtriangleup d' = c \cup d' = b'$, however $b'$ is disjoint from both $c$ and $d'$ so $b' = c = d' = \varnothing$. From this it is not hard to conclude that $a = c = \varnothing$ and, of course, $b = d$. $\square$

**Lemma 3.11.** *Let $p, q, r \in \beta G$ be ultrafilters such that $q + r = p$. If $\vec{y}$ is a sequence such that* $\mathrm{FS}(\vec{y}) \in p \cap r$*, then* $\mathrm{FS}^{\pm}(\vec{y}) \in q$*.*

*Proof.* The assumption is that

$$\{y \in G \big| -y + \mathrm{FS}(\vec{y}) \in r\} \in q,$$

so it suffices to show that every $y \in G$ which is such that $-y + \mathrm{FS}(\vec{y}) \in r$ will also satisfy that $y \in \mathrm{FS}^{\pm}(\vec{y})$. In order to argue that, notice first that for every such $y$ we have that $(-y + \mathrm{FS}(\vec{y})) \cap \mathrm{FS}(\vec{y}) \in r$, in particular this set is nonempty and so it is possible to find $x, z \in \mathrm{FS}(\vec{y})$ such that $-y + x = z$ which means that $y = x - z \in \mathrm{FS}^{\pm}(\vec{y})$. □

The following is the last lemma before we can actually state the proof of Theorem 3.8.

**Lemma 3.12.** *Suppose that $p \in G^*$ is a strongly summable ultrafilter and $q, r \in \beta G$ are such that $q + r = p$ and $\vec{x}$ is a sequence satisfying 2-uniqueness of sums with* $\mathrm{FS}(\vec{x}) \in p$*. If* $(\forall n < \omega)(\mathrm{FS}_n(\vec{x}) \in r)$*, then $r = q = p$.*

*Proof.* The first step for this proof is to notice that, under the stated hypotheses, we have that $\mathrm{FS}(\vec{x}) \in q$. It is certainly the case that $\mathrm{FS}^{\pm}(\vec{x}) \in q$ because of Lemma 3.11, hence $\{y \in \mathrm{FS}^{\pm}(\vec{x}) \big| -y + \mathrm{FS}(\vec{x}) \in r\} \in q$, so it suffices to show that this set is a subset of $\mathrm{FS}(\vec{x})$ in order to establish our claim. So let $y \in \mathrm{FS}^{\pm}(\vec{x})$ be such that $-y+\mathrm{FS}(\vec{x}) \in r$, and assume that $y = \sum_{n\in a} x_n - \sum_{n\in b} x_n$ with $a \cap b = \varnothing$. We let $m = \max(a\cup b)+1$ and note that $(-y+\mathrm{FS}(\vec{x})) \cap \mathrm{FS}_m(\vec{x}) \in r$, in particular this set is nonempty so we can take a $z \in \mathrm{FS}_m(\vec{x})$ such that $z + y \in \mathrm{FS}(\vec{x})$. Say that $z = \sum_{n\in c} x_n$, with $\min(c) \geq m > \max(a \cup b)$, then

$$z + y = \sum_{n\in a\cup c} x_n - \sum_{n\in b} x_n,$$

which immediately implies, by Lemma 3.10, that $b = \varnothing$ and so $y = \sum_{n\in a} x_n \in \mathrm{FS}(\vec{x})$.

Now for proving the current lemma, Theorem 3.6 ensures that it suffices to prove that $q = p$, so assume that this is not the case, i.e. $q \neq p$ and let $A \in p \setminus q$. Let $B = \mathrm{FS}(\vec{x}) \setminus A$ and pick a sequence $\vec{y}$ such that $p \ni \mathrm{FS}(\vec{y}) \subseteq \mathrm{FS}(\vec{x}) \cap A$. Define $a_n \in [\omega]^{<\omega} \setminus \varnothing$ by $y_n = \sum_{i\in a_n} x_i$ (notice that the $a_n$ must be pairwise disjoint). Now since $\mathrm{FS}(\vec{y}) \in p$, we have that

$$\{y \in \mathrm{FS}(\vec{x}) \big| -y + \mathrm{FS}(\vec{y}) \in r\} \in q,$$

in particular the set is nonempty and so we can pick a $y \in \mathrm{FS}(\vec{x})$ such that $-y + \mathrm{FS}(\vec{y}) \in r$. Say that $y = \sum_{i\in a} x_i$ and note that $(-y + \mathrm{FS}(\vec{y})) \cap B \in r$, so it is possible to choose a $z \in B$ with $y + z \in \mathrm{FS}(\vec{y})$. This means that, if $z = \sum_{i\in b} x_i$ then $a \cap b = \varnothing$ and $a \cup b$ is a union of some of the $a_i$, however $z \notin \mathrm{FS}(\vec{y})$ so one can conclude that for at least one $a_i$ we have $a_i \subseteq a \cup b$ and $a_i \cap a \neq \varnothing \neq a_i \cap b$. Thus $a_i \not\subseteq a$. Let $m = \max(a_i)+1$ and note that $(-y+\mathrm{FS}(\vec{y}))\cap\mathrm{FS}_m(\vec{x}) \in r$, in particular the set is nonempty and so we can choose a $w \in \mathrm{FS}(\vec{x})$ such that $y + w \in \mathrm{FS}(\vec{y})$. However, if $w = \sum_{j\in c} x_j$ with $\min(c) \geq m > \max(a_i)$, then $y + w = \sum_{j\in a\cup c} x_j$ which cannot be an element of $\mathrm{FS}(\vec{y})$ because $a_i \cap (a \cup c) \neq \varnothing$ but $a_i \not\subseteq (a \cup c)$. This is a contradiction, and we are done. □

We are now ready for proving the result that we have been slowly approaching.

*Proof of Theorem 3.8.* Let $p$ be a sparse ultrafilter such that $\mathrm{FS}(\vec{x}) \in p$ for some $\vec{x}$ satisfying 2-uniqueness of sums; and let $q, r \in G^*$ be such that $q + r = p$. There are two cases: if there exists a translate $x + r$ of $r$ (where $x \in G$) such that for some $m < \omega$ we have that $(\forall n \geq m)(\mathrm{FS}_n(\vec{x}) \in x + r)$, then Lemma 3.12 applied to $q - x, x + r$ and the sequence $\langle x_n \big| n \geq m\rangle$ shows that $q - x = x + r = p$, which means that $q = p + x$ and $r = -x + p$ and we are done. So the difficult case is when the opposite happens, namely for every $x \in G$ the translate $x + r$ does not contain $\mathrm{FS}_n(\vec{x})$ for infinitely many $n < \omega$.

Notice first of all that, since $\{y \in G \big| -y + \mathrm{FS}(\vec{x}) \in r\} \in q$, in particular there exists a $y \in G$ such that $\mathrm{FS}(\vec{x}) \in r + y$. We define $r' = r + y$ and $q' = q - y$, so that $q' + r' = p$; and we note that $q, r \in G + p$ if and only if $q', r' \in G + p$. Hence, from now on (switching to $q'$ and $r'$) we will assume that $\mathrm{FS}(\vec{x}) \in r$, and so by Lemma 3.11 we can also assume that $\mathrm{FS}^{\pm}(\vec{x}) \in q$. Now the assumption for this case implies that for some $n < \omega$ we have that $\mathrm{FS}_{n+1}(\vec{x}) \notin r$, letting $n_0$ be the least such $n$ yields that $x_{n_0} + \mathrm{FS}_{n_0+1}(\vec{x}) \in r$. Recursively, if we have picked $n_0 < n_1 < \cdots n_k$ such that $x_{n_0} + \cdots + x_{n_k} + \mathrm{FS}_{x_{n_k}+1}(\vec{x}) \in r$, the relevant assumption allows us to let $n_{k+1}$ be the least $n > n_k$ satisfying that $\mathrm{FS}_{n+1}(\vec{x}) \notin -(x_{n_0} + \cdots + x_{n_k}) + r$, and we will then have that $x_{n_0} + \cdots + x_{n_k} + x_{n_{k+1}} + \mathrm{FS}_{n_{k+1}+1}(\vec{x}) \in r$. This way we are recursively constructing an increasing sequence $\langle n_k \big| k < \omega \rangle$ such that, for all $n < \omega$,

$$\left( \sum_{i \leq n} x_{k_i} \right) + \mathrm{FS}_{k_n+1}(\vec{x}) \in r.$$

It is not hard to show that we also have, for every $n < \omega$, that $-\left(\sum_{i \leq n} x_{k_i}\right) + \mathrm{FS}^{\pm}(\vec{x}) \in q$, but we will not use that fact for the current argument.

**Claim 3.1.**

$$\mathrm{FS}(\langle x_{k_n} \big| n < \omega \rangle) \in p$$

*Proof of Claim.* Without loss of generality assume that $\{k_n \big| n < \omega\}$ is coinfinite (otherwise there is nothing to prove). We will first of all argue that $A = \mathrm{FS}(\langle x_i \big| i \notin \{k_n \big| n < \omega\}\rangle) \notin p$, since if $A$ belonged to $p$ then there would be a $y \in \mathrm{FS}^{\pm}(\vec{x})$ (there would actually be $q$-many of them) such that $-y + A \in r$. Say that $y = \sum_{i \in a} x_i - \sum_{i \in b} x_i$ with $a \cap b = \varnothing$ and let $n$ be such that $k_n > \max(a \cup b)$. We can now pick ($r$-many) $z = \sum_{i \leq n} x_{k_i} + \sum_{i \in c} x_i$, with $\min(c) > k_n$, such that $y + z \in A$. However,

$$y + z = \sum_{i \in a} x_i - \sum_{i \in b} x_i + \sum_{i \leq n} x_{k_i} + \sum_{i \in c} x_i \in A \subseteq \mathrm{FS}(\vec{x})$$

hence by Lemma 3.10, we must have that $b \subseteq \{k_i \big| i \leq n\}$ and $a \cap \{k_i \big| i \leq n\} = \varnothing$, therefore $y + z = \sum_{i \in d} x_i$ where $d = a \cup c \cup \{k_i \big| i \leq n\} \setminus b$. But since $k_n > \max(b)$, this means that $y + z$ cannot be an element of $A$ because $k_n \in d$.

We now suppose that the conclusion of the claim is false. Given what we just proved in the previous paragraph, we have now that

$$B = \left\{ \sum_{i \in a} x_i \in \mathrm{FS}(\vec{x}) \middle| a \cap \{k_n \big| n < \omega\} \neq \varnothing \text{ and } a \nsubseteq \{k_n \big| n < \omega\} \right\} \in p,$$

so we can pick a sequence $\vec{y}$ satisfying that $p \ni \mathrm{FS}(\vec{y}) \subseteq \mathrm{FS}(\vec{x}) \cap B$. Under these conditions, if we let $a_n$ be given by $y_n = \sum_{i \in a_n} x_i$, then the $a_n$ will be pairwise disjoint, and each of them will contain some $k_j$, but at the same time we will have that $a_n \nsubseteq \{k_n \big| n < \omega\}$. Now, there are $q$ many $y \in \mathrm{FS}^{\pm}(\vec{x})$ such that $-y + \mathrm{FS}(\vec{y}) \in r$, and if such a $y$ is written as $y = \sum_{i \in a} x_i - \sum_{i \in b} x_i$ for $a \cap b = \varnothing$, then picking any $m$ big enough so that $\min(a_m) > \max(a \cup b)$ and picking $n$ such that $k_n > \max(a_m)$, we have that for $r$ many $z \in \left(\sum_{i \leq n} x_{k_i}\right) + \mathrm{FS}_{k_n+1}(\vec{x})$ it will be the case that $y + z \in \mathrm{FS}(\vec{y})$. However, $z$ must be written as $z = \sum_{i \leq n} x_{k_i} + \sum_{i \in c} x_i$ with $\min(c) > k_n$, so that

$$y + z = \sum_{i \in a} x_i - \sum_{i \in b} x_i + \sum_{i \leq n} x_{k_i} + \sum_{i \in c} x_i,$$

which makes it impossible for $y + z$ to be an element of $\mathrm{FS}(\vec{y})$, since by choice of $n$, there will be a $j < n$ with $k_j \in a_m$ and so the composition of $y + z$ as a finite sum from the sequence $\vec{x}$ will include some elements of $a_m$, but not all of them since

$a_m \nsubseteq \{k_n \big| n < \omega\}$. $\square$

We will now finally make use of the hypothesis that $p$ is sparse. By Claim 3.1, together with the sparseness of $p$, it is possible to find a sequence $\vec{y}$ and a subsequence of it $\vec{z}$ such that $\{y_n \big| n < \omega\} \setminus \{z_n \big| n < \omega\}$ is infinite, $\mathrm{FS}(\vec{z}) \in p$ and $\mathrm{FS}(\vec{y}) \subseteq \mathrm{FS}(\langle x_{k_n} \big| n < \omega\rangle)$. If we write each $z_n = \sum_{i \in a_n} x_{k_i}$ and we let $M = \bigcup\limits_{n<\omega} a_n$, it is not hard to see that $M$ must be a coinfinite subset of $\omega$, and $\mathrm{FS}(\langle x_{k_n} \big| n \in M\rangle) \in p$ (since this set is a superset of $\mathrm{FS}(\vec{z})$). This in turn implies that

$$\{y \in \mathrm{FS}^{\pm}(\vec{x}) \big| -y + \mathrm{FS}(\langle x_{k_n} \big| n \in M\rangle) \in r\} \in q,$$

in particular the above set contains at least one element $y$, which we can write in the form $y = \sum_{i \in a} x_i - \sum_{i \in b} x_i$ for $a \cap b = \varnothing$. Let $m \in \omega \setminus M$ be such that $k_m > \max(a \cup b)$ and notice that

$$\left(-y + \mathrm{FS}(\langle x_{k_n} \big| n \in M\rangle)\right) \cap \left(\left(\sum_{i \leq m} x_{k_i}\right) + \mathrm{FS}_{k_m+1}(\vec{x})\right) \in r,$$

in particular the above set is nonempty and so we can pick some $w$, with $w = \sum_{i \leq n} x_{k_i} + \sum_{i \in c} x_i$ where $\min(c) > k_m$, such that $y + w \in \mathrm{FS}(\langle x_{k_n} \big| n \in M\rangle) \subseteq \mathrm{FS}(\vec{x})$. However,

$$y + w = \sum_{i \in a} x_i - \sum_{i \in b} x_i + \sum_{i \leq n} x_{k_i} + \sum_{i \in c} x_i$$

where $\min(c) > k_m > \max(a \cup b)$. Hence by Lemma 3.10 we must have that $b \subseteq \{k_i \big| i \leq m\}$ and $a \cap \{k_i \big| i \leq m\} = \varnothing$, so that $y + w = \sum_{i \in d} x_i$, where $d = a \cup c \cup \{k_i \big| i \leq m\} \setminus b$. But this is a contradiction since $k_m \in d$ but $k_m \notin M$. $\square$

It will be shown in Section 4.1 that the existence of strongly summable ultrafilters on abelian groups (in particular, the existence of ultrafilters satisfying the hypothesis of Theorem 3.8) cannot be proved in $\mathsf{ZFC}$. It is still open whether one can prove in $\mathsf{ZFC}$ alone that there exist idempotent ultrafilters (on some abelian group $G$) satisfying the trivial sums property.

We now turn to the noncommutative analogue of these notions and results.

**Definition 3.13.** A sequence $\vec{x}$ in a semigroup $S$ satisfies the **ordered uniqueness of products** if it satisfies the uniqueness of products, and additionally, whenever $a, b \in [\omega]^{<\omega}$ are such that

$$\left(\prod_{n\in a} x_n\right) * \left(\prod_{n\in b} x_n\right) \in \mathrm{FP}(\vec{x}),$$

it must be the case that $\max(a) < \min(b)$

This notion is clearly only worth looking at for noncommutative semigroups, as no commutative semigroup can possibly have any sequence of elements satisfying the ordered uniqueness of products. But, for example, if $S$ is the free semigroup on the (countably many) generators $\{s_n \big| n < \omega\}$, then it is not terribly hard to check that the sequence $\langle s_n \big| n < \omega\rangle$ of generators of $S$ satisfies the ordered uniqueness of products. This notion is important because of the following reason.

**Lemma 3.14.** *Let $S$ be some semigroup and let $p \in S^*$ be a strongly productive ultrafilter such that for some sequence $\vec{x}$ satisfying the ordered uniqueness of products* *we have that* $\mathrm{FP}(\vec{x}) \in p$*. Then* $p$ *is multiplicatively isomorphic to an ordered union ultrafilter.*

*Proof.* Reasoning as in the proof of Theorem 3.3, it is straightforward to verify that the mapping

$$\varphi : \mathrm{FP}(\vec{x}) \longrightarrow \mathbb{B}$$

$$\prod_{n \in a} x_n \longmapsto a$$

will map $p$ to an ordered union ultrafilter. □

We now think about the free group $G$ on the countably many generators $\{s_n \big| n < \omega\}$. This guy contains the free semigroup $S$ on the same generators as a subsemigroup. L. Legette [27, Definition 3.2] defined a **very strongly productive ultrafilter** to be a strongly productive ultrafilter on $G$ wich has a base of sets of the form $\mathrm{FP}(\vec{x})$ where $\vec{x}$ is a product subsystem of $\vec{s}$. It is easy to see that this is equivalent to just demanding that $p$ is a strongly productive ultrafilter on $G$ such that $\mathrm{FP}(\vec{s}) \in p$. Lemma 3.14 implies that every very strongly productive ultrafilter on $S$ is multiplicatively isomorphic to some ordered union ultrafilter.

One of the reasons why this notion is important is the result [27, Theorem 3.10] that if $p \in S^*$ is very strongly productive and $q, r \in \beta S$ are such that $q * r = p$, then $q = r = p$. If we allow $q$ and $r$ to be elements of the larger semigroup $\beta G$, we need to pay with a stronger hypothesis in order to keep the same thesis (although admittedly it is not clear at all to whom we are paying, nor with what currency).

**Definition 3.15.** Let $S$ be a non-commutative semigroup.

(i) We say that an idempotent $p*p = p \in S^*$ has the **trivial products property** if, whenever $q, r \in S^*$ are such that $q * r = p$, there must exist an $x \in S$ such that $q = p * x$ and $r = x^{-1} * p = \{x^{-1} * A \big| A \in p\}$.

(ii) We say that a strongly productive ultrafilter $p \in S^*$ is **sparse** if for every $A \in p$ there exists a sequence $\vec{x}$ of elements of $S$ such that $\mathrm{FP}(\vec{x}) \subseteq A$ and an infinite co-infinite subset $M \subseteq \omega$ such that $\mathrm{FP}(\langle x_n \big| n \in M\rangle) \in p$.

Note that the previous definition is different from the simple translation of Definition 3.7. In particular, for a group $G$, the (principal ultrafilter generated by the) identity element of $G$ is sparse if $G$ is non-abelian but not if $G$ is abelian. However, this apparent anomaly vanishes when we restrict our attention to very strongly productive ultrafilters on the free semigroup.

We close this section by stating without proof the following result, due to N. Hindman and L. Jones [18, Theorem 3.10]. It is worth noting that its proof is quite similar to the proof of Theorem 3.8 presented here, with the caveat that it becomes necessary to do some modifications to take into account the noncommutativity of the situation. In fact, it would probably be more accurate to say that the proof of

Theorem 3.8 presented here borrows lots of ideas from the proof of the following theorem that appears in [18], although it does introduce some modifications in order to adapt to the commutativity of that situation.

**Theorem 3.16.** *Let $p$ be a sparse very strongly productive ultrafilter on the free semigroup $S$. Then $p$ has the trivial products property in the free group $G$.*

## 3.2 Sparseness for Ultrafilters on the Boolean Group

Our study of sparse ultrafilters will focus first on the Boolean group $\mathbb{B}$. Given that $\mathrm{FS}(\vec{x}) = \mathrm{F}\triangle(\mathrm{ran}(\vec{x}))$ for every sequence $\vec{x}$ of elements of $\mathbb{B} = [\omega]^{<\omega}$, we can conclude that an ultrafilter $p \in \mathbb{B}^*$ is sparse if and only if for every $A \in p$ there exists a linearly independent set $X$ and an infinite co-infinite subset of it $Y \subseteq X$ such that $\mathrm{F}\triangle(Y) \in p$ and $\mathrm{F}\triangle(X) \subseteq A$. Although the notion of sparse seems at first sight stronger than the notion of strongly summable, we can right away establish that this is not so for the Boolean group.

**Theorem 3.17.** *Every strongly summable ultrafilter on $\mathbb{B}$ is sparse.*

*Proof.* Let $p \in \mathbb{B}^*$ be a strongly summable ultrafilter, and let $A \in p$. Because of strong summability, there is an infinite linearly independent $Z$ such that $p \ni \mathrm{F}\triangle(Z) \subseteq A$. We claim that there is a $B \in p$ such that for some infinite $W \subseteq Z$, $\mathrm{F}\triangle(W) \cap B = \varnothing$. Notice that the result follows easily from the claim: just pick a linearly independent $Y$ such that $p \ni \mathrm{F}\triangle(Y) \subseteq B \cap \mathrm{F}\triangle(Z)$, and let $X = Y \cup W$. Then it is straightforward to prove that $X$ is linearly independent, since so are $Y$ and $W$, and $\mathrm{F}\triangle(W)$ is disjoint from $\mathrm{F}\triangle(Y)$. Since $X \setminus Y = W$ we also have that $|X \setminus Y| = \omega$; and since $Y, W \subseteq \mathrm{F}\triangle(Z)$, we will have that $\mathrm{F}\triangle(X) \subseteq \mathrm{F}\triangle(Z) \subseteq A$ and we are done.

Thus we now proceed to prove our claim, since that will immediately establish the theorem. In order to do that, let $Z'$ be an infinite co-infinite subset of $Z$. Let

$$B_0 = \{w \in \mathrm{F}\triangle(Z) | \mathrm{supp}_Z(w) \cap Z' \neq \varnothing\},$$

and

$$B_1 = \mathrm{F}\triangle(Z) \setminus B_0 = \{w \in \mathrm{F}\triangle(Z) | \mathrm{supp}_Z(w) \cap Z' = \varnothing\}.$$

There is $i \in 2$ such that $B_i \in p$. If $B_0 \in p$ then we let $W = Z \setminus Z'$; otherwise if $B_1 \in p$ we let $W = Z'$. In any case it is easy to see that $\mathrm{F}\triangle(W) \cap B_i = \varnothing$. □

In particular, union and ordered union ultrafilters are sparse. In general, strongly summable ultrafilter on the Boolean group are particularly well-behaved, as the following result of Protasov's [33, Corollary 4.4] shows.

**Theorem 3.18.** *Every strongly summable ultrafilter on the Boolean group $\mathbb{B}$ has the trivial sums property.*

At the end of the day, this chapter is all about proving that most strongly productive ultrafilters (especially strongly summable ultrafilters on abelian groups) are sparse. Hence it makes sense to try and establish a simple condition that will ensure sparseness on a given strongly productive ultrafilter. For this we need some more theory.

**Definition 3.19.**

(i) Given a set $M \subseteq \omega$, a **block** of $M$ is a maximal interval contained in $M$. For example, if $M = \{0, 1, 2, 6, 7, 10, 14, 15, 16, 17\}$ then the blocks of $M$ are the sets $\{0, 1, 2\}, \{6, 7\}, \{10\}, \{14, 15, 16, 17\}$.

(ii) If $M$ is finite, then a **gap** of $M$ is a maximal interval contained in $\max(M) \setminus M$. For example, if $M$ is as above then its gaps are $\{3, 4, 5\}, \{8, 9\}, \{11, 12, 13\}$.

(iii) Given a finite set $a \in \mathbb{B}$ we let $\mathrm{N.\,B.}(a)$ denote the number of blocks that $a$ has. For example, if $M$ is as above then $\mathrm{N.\,B.}(M) = 4$. This defines a function $\mathrm{N.\,B.} : \mathbb{B} \longrightarrow \omega$.

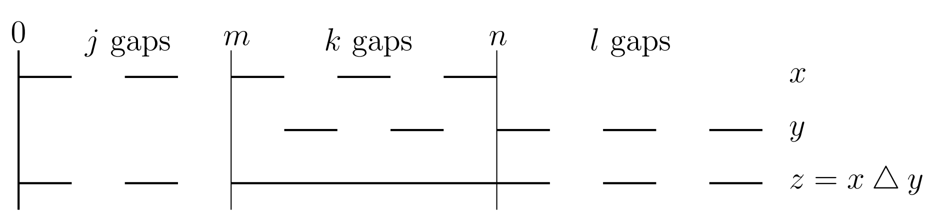
(iv) We could analogously define the number of gaps of a finite set $a \in \mathbb{B}$, but we will instead just note that this number equals $\mathrm{N.\,B.}(a)$ if $0 \notin a$ and $\mathrm{N.\,B.}(a) - 1$ otherwise.

The following lemma is due to Hindman, Steprāns and Strauss [20, Theorem 2.5]. We present our own proof here.

**Lemma 3.20.** *Whenever* $X \subseteq \mathbb{B}$ *is a pairwise disjoint family such that* $(\exists i \in 2)(\forall x \in \mathrm{F}\triangle(X))(\mathrm{N.B.}(x) \equiv i \mod 2)$*, then* $\bigcup X$ *is coinfinite.*

*Proof.* We prove the case $i = 0$ and emphasize that the case $i = 1$ is entirely analogous. So assume by way of contradiction that $X$ is a pairwise disjoint family such that $\bigcup X$ is cofinite and $\mathrm{N.B.}(x)$ is even for every $x \in \mathrm{F}\triangle(X)$. We first notice that it is possible to assume without loss of generality that $0 \in \bigcup X$ (just pick the $x \in X$ with least minimum and replace it by $x \cup \min(x)$, which shall not change the number of blocks of any element of $\mathrm{F}\triangle(X)$).

Now let $m = \max(\omega \setminus \bigcup X)$ and, for every $i < m$ such that $i \in \bigcup X$ pick an $x_i \in X$ such that $i \in x_i$. Note that $x = \bigcup_{\substack{i<m \\ i \in \bigcup X}} x_i \in \mathrm{F}\triangle(X)$. Now let $n = \max(x)$ and, for every $j < n$ such that $j \in \bigcup X \setminus x$, pick a $y_j \in X$ such that $j \in y_j$ (notice that every such $y_j$ must be disjoint from $x$). Let $y = \bigcup_{\substack{j<n \\ j \in \bigcup X}} y_j \in \mathrm{F}\triangle(X)$. Then $x$ and $y$ are mutually disjoint and $z = x \triangle y = x \bigcup y \in \mathrm{F}\triangle(X)$. Consequently $\mathrm{N.B.}(x), \mathrm{N.B.}(y), \mathrm{N.B.}(z)$ are all even.




Let $j = \mathrm{N.B.}(\omega \setminus \bigcup X)$ (which is the number of gaps that $x$ has below $m$), let $k = \mathrm{N.B.}(x) - j - 1$ (which counts the number of gaps of $x$ that are located between $m$ and $n$, since $x$ has $\mathrm{N.\,B.}(x)-1$ gaps); and let $l$ be the number of gaps of $y \cup (n+1)$ (the number of gaps of $y$ that are past $n$). We see (e.g. in the picture) that, for each gap of $y$ that is below $n$, we can count exactly one block of $x$ that is between $m$ and $n$, hence there are $k+1$ such gaps and so the total number of gaps of $y$ is $l+k+1$. Since $0 \notin y$, then $y$ has the same number of gaps as it has of blocks, thus $l+k+1 = \mathrm{N.\,B.}(y)$ which is an even number and so $k$ and $l$ have to be of opposite parity. Now notice that $j$ and $k$ are of opposite parity (since $\mathrm{N.\,B.}(x)$ is even), hence $j$ and $l$ have the same parity and so $j+l$ is even. On the other hand, $z = x \triangle y$ has $j+l$ gaps (an even number), but at the same it has (since $0 \in z$) $\mathrm{N.\,B.}(z)-1$ gaps (an odd number), which is a contradiction. □

The following theorem originally appeared in [20, Theorem 2.6], and can be traced back to [22, Theorem 4].

**Theorem 3.21.** *If $p$ is a union ultrafilter and $M \subseteq \omega$ then there exists an $A \in p$ such that $M \setminus \bigcup A$ is infinite.*

*Proof.* We first treat the case where $M = \omega$. Just choose $i \in 2$ such that $A_i = \{x \in \mathbb{B} \big| \mathrm{N.\,B.}(x) \equiv i \mod 2\} \in p$ and let $X$ be a pairwise disjoint family such that $p \ni \mathrm{F}\triangle(X) \subseteq A_i$. Declaring $A = \mathrm{F}\triangle(X)$ does the job because of Lemma 3.20.

Now for arbitrary $M \subseteq \omega$, if $[\omega \setminus M]^{<\omega} \in p$ then we are done. Otherwise $B = \{a \in \mathbb{B} \big| a \cap M \neq \varnothing\} \in p$, and let us note that $q = \{\{a \cap M \big| a \in A\} \big| A \in p \upharpoonright B\}$

is a union ultrafilter on $[M]^{<\omega}$, so if we let $M = \{m_n \big| n < \omega\}$ be any enumeration of $M$ then the function $\{m_{n_1}, \dots, m_{n_k}\} \longmapsto \{n_1, \dots, n_k\}$ will map $q$ to another union ultrafilter which must have an element $C$ such that $\bigcup C$ is coinfinite in $\omega$. There must be an $A \in p \restriction B \subseteq p$ such that the preimage of $C$ under this mapping is exactly $\{a \cap M \big| a \in A\}$, and it must be the case that $M \setminus \bigcup A$ is infinite (since $\omega \setminus \bigcup C$ is infinite). Thus, we are done. $\square$

At this point, in order to apply Theorem 3.21, we need to distinguish between the commutative and the noncommutative case. First the commutative one.

**Theorem 3.22.** *Let $p$ be a strongly summable ultrafilter on some abelian semigroup $S$. If $p$ is additively isomorphic to a union ultrafilter then $p$ is sparse.*

*Proof.* If $p$ is additively isomorphic to some union ultrafilter, by Proposition 1.11 we can pick a sequence $\vec{x}$ satisfying uniqueness of finite sums such that $\mathrm{FS}(\vec{x}) \in p$, and such that the mapping $\varphi$ given by $\varphi(\sum_{n\in a} x_n) = a$ maps $p$ to a union ultrafilter $q$. Let $A \in p$, and let $X$ be pairwise disjoint such that $q \ni \mathrm{F}\triangle(X) \subseteq \varphi[A \cap \mathrm{FS}(\vec{x})]$. Now let $M = \bigcup X$. Since $q$ is a union ultrafilter, Theorem 3.21 ensures that there is $B \in q$ such that $M \setminus \bigcup B$ is infinite. Without loss of generality we can assume $B \subseteq \mathrm{F}\triangle(X)$, so that $\bigcup B$ is a coinfinite subset of $M$. Grab a pairwise disjoint family $Y$ such that $q \ni \mathrm{F}\triangle(Y) \subseteq B$, then $\bigcup Y$ is a coinfinite subset of $M = \bigcup X$ and thus there are infinitely many $x \in X$ that do not intersect $\bigcup Y$ (because $Y \subseteq \mathrm{F}\triangle(X)$

and $X$ is a pairwise disjoint family, so if $x \in X$ intersects $\bigcup Y$ then $x \subseteq \bigcup Y$). Thus if we let $Z = \{x \in X \big| x \cap \bigcup Y = \varnothing\} \cup Y$ then $Z$ is a pairwise disjoint family and $\mathrm{F}\triangle(Z) \subseteq \mathrm{F}\triangle(X) \subseteq \varphi[A \cap \mathrm{FS}(\vec{x})]$. Enumerate $Z = \{z_n \big| n < \omega\}$ in such a way that $Y = \{z_{2n} \big| n < \omega\}$ and $\{x \in X \big| x \cap \bigcup Y = \varnothing\} = \{z_{2n+1} \big| n < \omega\}$. Then let $\vec{w}$ be given by $w_n = \sum_{i \in z_n} x_i$. We get that $\mathrm{FS}(\vec{w}) = \varphi^{-1}[\mathrm{F}\triangle(Z)] \subseteq A$, and if $\vec{y}$ is the subsequence of even elements of $\vec{w}$, then we will have that $|\{w_n \big| n < \omega\} \setminus \{y_n \big| n < \omega\}|$ is infinite and $\mathrm{FS}(\vec{y}) = \varphi^{-1}[\mathrm{F}\triangle(Y)] \in p$. □

From this we can easily conclude [20, Theorem 3.2.] which will be instrumental in what follows.

**Corollary 3.23.** *Let $p$ be a strongly summable ultrafilter on some abelian group $G$ such that there exists a sequence $\vec{x}$ satisfying the 2-uniqueness of sums with* $\mathrm{FS}(\vec{x}) \in p$. *Then $p$ is sparse.* □

**Corollary 3.24.** *Every strongly summable ultrafilter on $\mathbb{N}$ is sparse.* □

We will also be able to deduce the noncommutative equivalent of Corollary 3.23.

**Corollary 3.25.** *Let $p$ be a strongly productive ultrafilter on some semigroup $S$. If $p$ is multiplicatively isomorphic to an ordered union ultrafilter then $p$ is sparse.*

*Proof.* If $p$ is multiplicatively isomorphic to an ordered union ultrafilter, by Proposition 1.11 we can assume that there is a sequence $\vec{x}$ such that $\mathrm{FP}(\vec{x}) \in S$ and the function $\varphi : \mathrm{FP}(\vec{x}) \longrightarrow \mathbb{B}$ given by $\varphi(\prod_{n\in a} x_n) = a$ is the witnessing multiplicatively isomorphism mapping $p$ to an ordered union ultrafilter $q$. Given any $A \in p$, observe that $\varphi[A] \in q$, so we can pick an ordered family $Z$ in $\mathbb{B}$ such that $q \ni \mathrm{F}\triangle(Z) \subseteq \varphi[A]$. Moreover by Theorem 3.21 there is an element $B \in q$, which (without loss of generality) is a subset of $\mathrm{F}\triangle(Z)$ such that $\bigcup Z \setminus \bigcup B$ is infinite. Let $Z = \{z_n \big| n < \omega\}$ be the increasing enumeration of $Z$ (i.e. $\max(z_n) < \min(z_{n+1})$ for all $n < \omega$) and denote by $M = \{n < \omega \big| z_n \subseteq \bigcup B\}$. Observe that $\bigcup_{n\in M} z_n = \bigcup B$ and $B \subseteq \mathrm{F}\triangle(\{z_n \big| n \in M\})$. In particular $M$ is a coinfinite subset of $\omega$ and $\mathrm{F}\triangle(\{z_n \big| n \in M\}) \in q$. Therefore, if we define the sequence $\vec{y}$ by $y_n = \prod_{i\in z_n} x_i$, then we will get that $\mathrm{FP}(\vec{y}) \subseteq A$ and $\mathrm{FP}(\langle y_n \big| n \in M\rangle) \in p$, hence $p$ is sparse strongly productive. $\square$

We will use Corollary 3.23 at the end of next section in order to extract some consequences for strongly summable ultrafilters on abelian groups. However, Corollary 3.25 can be used right away to obtain a very interesting property of very strongly productive ultrafilters on the free semigroup.

**Corollary 3.26.** *Every very strongly productive ultrafilter on the free semigroup $S$ is sparse and hence it has the trivial products property.* $\square$

## 3.3 Sparseness and Trivial Sums on Abelian Groups

The main result of this section tells us that almost all strongly summable ultrafilters on abelian groups have FS-sets generated from sequences that satisfy 2-uniqueness of finite sums. As a consequence of that, almost all strongly summable ultrafilters on abelian groups are essentially (that is, additively isomorphic to) union ultrafilters (because of Theorem 3.3), and this helps solve [20, Questions 4.11 and 4.12]. More precisely, we have the following theorem and corollary.

**Theorem 3.27.** *Let $G$ be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter such that*

$$\{x \in G \big| o(x) = 2\} \notin p.$$

*Then, there exists a sequence $\vec{x}$ of elements of $G$ satisfying the 2-uniqueness of finite sums such that* $\mathrm{FS}(\vec{x}) \in p$.

**Corollary 3.28.** *Let $G$ be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter such that*

$$\{x \in G \big| o(x) = 2\} \notin p.$$

*Then $p$ is additively isomorphic to some union ultrafilter.*

In order to prove this result, we will need to break the proof down into several subcases.

**Lemma 3.29.** *Let $G$ be an abelian group, and let $X = \{x \in G \big| o(x) = 4\}$. If $\vec{x}$ is a sequence of elements of $G$ such that* $\mathrm{FS}(\vec{x}) \subseteq X$*, then $\vec{x}$ must satisfy 2-uniqueness of finite sums.*

*Proof.* Assume that $\vec{x}$ is such that $\mathrm{FS}(\vec{x}) \subseteq X$. By Proposition 3.2, in order to prove that $\vec{x}$ satisfies 2-uniqueness of finite sums, it suffices to show that whenever $a, b, c, d$ are such that $a \cap b = \varnothing = c \cap d$ and

$$2\sum_{n\in a} x_n + \sum_{n\in b} x_n = 2\sum_{n\in c} x_n + \sum_{n\in d} x_n,$$

then $a = c$ and $b = d$. Now for each $n \in b \cap d$ we can cancel the term $x_n$ from both sides of the previous equation; and similarly for each $n \in a \cap c$ we can cancel the term $2x_n$ from both sides of the equation, which thus becomes

$$2\sum_{n\in a'} x_n + \sum_{n\in b'} x_n = 2\sum_{n\in c'} x_n + \sum_{n\in d'} x_n, \tag{3.1}$$

where $a' = a \setminus (a \cap c)$, $b' = b \setminus (b \cap d)$, $c' = c \setminus (a \cap c)$ and $d' = d \setminus (b \cap d)$. Since $b'$ is disjoint from $d'$, Equation (3.1) yields

$$\sum_{n\in b'\cup d'} x_n = \sum_{n\in b'} x_n + \sum_{n\in d'} x_n = -2\sum_{n\in a'} x_n + 2\sum_{n\in c'} x_n + 2\sum_{n\in d'} x_n,$$

where each of the terms from the right-hand side is either the identity (if the corresponding sum happens to be an empty sum) or has order 2 (because if the corresponding sum is nonempty then it has order 4), so the right-hand side of the previous equation is either the identity or has order 2. If $b' \cup d'$ was nonempty, the left-hand side of this equation would be a legitimate element of $\mathrm{FS}(\vec{x})$, thus of order 4, and this is impossible. Hence we must have that $b' = d' = \varnothing$, which means that $b = b \cap d = d$. Therefore (3.1) becomes

$$2 \sum_{n \in a'} x_n = 2 \sum_{n \in c'} x_n,$$

which in turn implies that

$$2 \left( \sum_{n \in a'} x_n - \sum_{n \in c'} x_n \right) = 0,$$

and this means that the element $x = \sum_{n \in a'} x_n - \sum_{n \in c'} x_n$ is either the identity, or of order 2. Now since $a'$ is disjoint from $c'$, we get

$$\sum_{n \in a' \cup c'} x_n = \sum_{n \in a'} x_n + \sum_{n \in c'} x_n = x + 2 \sum_{n \in c'} x_n.$$

Again, each term on the right-hand side is either the identity or has order 2, so the whole right-hand side is either the identity or of order 2. So, arguing as we did before, we conclude that $a' = c' = \varnothing$, which means that $a = a \cap c = d$, and therefore $\vec{x}$ satisfies 2-uniqueness of finite sums. $\square$

If $G$ is any abelian group, and $p \in G^*$ is strongly summable, then there must be a countable subgroup $H$ such that $H \in p$ (e.g. take any FS set in $p$ because of strong summability, and then let $H$ be the subgroup generated by such FS set), and certainly the restricted ultrafilter $p \restriction H = p \cap \mathfrak{P}(H)$ will also be strongly summable. If we prove that $p \restriction H$ contains a set of the form $\mathrm{FS}(\vec{x})$ for a sequence $\vec{x}$ satisfying

2-uniqueness of finite sums, then certainly so does $p$ itself, because $p$ is just the ultrafilter generated in $G$ by $p \upharpoonright H$ and in particular $p \upharpoonright H \subseteq p$. Hence in order to prove Theorem 3.27, it suffices to consider only countable abelian groups $G$, and we will do so in the remainder of this section. This simplifies matters because (as discussed in the Introduction) every countable abelian group $G$ can be embedded in the group $\bigoplus_{n<\omega} \mathbb{T}$, and any strongly summable ultrafilter on $G$ generates a strongly summable ultrafilter on $\bigoplus_{n<\omega} \mathbb{T}$, which will contain some $\mathrm{FS}(\vec{x})$ for some $\vec{x}$ satisfying 2-uniqueness of sums if and only if the original ultrafilter does. Thus from now on we will only deal with $G = \bigoplus_{n<\omega} \mathbb{T}$.

**Definition 3.30.** In the remainder of this section, we will denote by

$$Q(G) = \{x \in G \big| o(x) \in \{1, 2, 4\}\}$$

for all $n < \omega$. Note that $Q(G)$ is the subgroup of $G$ consisting of those elements satisfying that $\pi_n(x) \in \left\{0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}\right\}$. Thus for $x \notin Q(G)$ then there is an $n < \omega$ such that $\pi_n(x) \notin \left\{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right\}$. We will denote the least such $n$ by $\rho(x)$.

The following theorem subsumes as a particular case a theorem of Hindman, Steprāns and Strauss [20, Theorem 4.5].

**Theorem 3.31.** *Let $G$ be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter. If*

$$\left\{x \in G \setminus Q(G) \middle| \pi_{\rho(x)}(x) \notin \left\{\frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{3}{8}\right\}\right\} \in p,$$

*then there exists a set $X \in p$ such that for every sequence $\vec{x}$ of elements of $\bigoplus_{n<\omega}\mathbb{T}$, if $\mathrm{FS}(\vec{x}) \subseteq X$ then $\vec{x}$ must satisfy 2-uniqueness of finite sums.*

*Proof.* Note that, if $p$ is as in the hypotheses, then $p$ must contain as an element one of the sets

$$\left\{x \in G \setminus Q(G) \middle| \pi_{\rho(x)}(x) \in I\right\}$$

where $I$ is either $\left(0, \frac{1}{8}\right)$, $\left(\frac{1}{8}, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{3}{8}\right)$, $\left(\frac{3}{8}, \frac{1}{2}\right)$, $\left(-\frac{1}{8}, 0\right)$, $\left(-\frac{1}{4}, -\frac{1}{8}\right)$, $\left(-\frac{3}{8}, -\frac{1}{4}\right)$, or $\left(-\frac{1}{2}, -\frac{3}{8}\right)$. We will assume that $I = \left(0, \frac{1}{8}\right)$, the proof in all other cases being entirely analogous. We note that we can partition $I = I_0 \cup I_1 \cup I_2$, where

$$I_i = \bigcup_{m\in\mathbb{N}} \left[\frac{1}{2^{3m+1+i}}, \frac{1}{2^{3m+i+2}}\right)$$

This is, $I_i$ consists of those real numbers $t \in \left(0, \frac{1}{8}\right)$ whose first nonzero digit, in the binary expansion, lies in a position that is $\equiv i \mod 3$ (note that this first nonzero digit lies at least in the fourth position since $t < \frac{1}{8}$). Note that if $r, t \in I_i$ and $r + t \in I_i$, then $r$ cannot have its first nonzero digit in the same position as $t$ does so if $r > t$ then actually $r > 4t$.

We grab an $i \in 3$ such that $X = \{x \in G \setminus Q(G) \big| \pi_{\rho(x)}(x) \in I_i\} \in p$, and we claim that $X$ is as desired in the conclusion of the theorem. So assume that $\vec{x}$ is such that $\mathrm{FS}(\vec{x}) \subseteq X$. For each $i < \omega$, we let $M_i = \{n < \omega \big| \rho(x_n) = i\}$ and $M = \{i < \omega \big| M_i \neq \varnothing\}$. The observation from the previous paragraph implies that if $n, m \in M_i$ then $\pi_i(x_n) \neq \pi_i(x_m)$ and actually one of these numbers is greater than

4 times the other. Thus, rearranging the sequence $\vec{x}$ if necessary, we can always assume that $n < m$ and $n, m \in M_i$ implies that $\pi_i(x_n) > 4\pi_i(x_m)$. In fact, from this it follows that, for any $n \in M_i$, we have that

$$\pi_i(x_n) > 3 \sum_{\substack{n<m \\ m\in M_i}} \pi_i(x_m). \tag{3.2}$$

Now notice that, if $i < j$ and $i \in M$ and $n \in M_j$, then $\pi_i(x_n) = 0$; since this is the only way that $\pi_i(x_n + x_m) \in I$ if $m \in M_i$. Hence, whenever we have sets $a, b \in [\omega]^{<\omega} \setminus \{\varnothing\}$ and functions $\varepsilon : a \longrightarrow \{1, 2\}$, $\delta : b \longrightarrow \{1, 2\}$; if

$$\sum_{n\in a} x_n = \sum_{n\in b} x_n,$$

letting $i$ be least such that $(a \cup b) \cap M_i \neq \varnothing$ we get that

$$\sum_{n\in a\cap M_i} \pi_i(x_n) = \sum_{n\in b\cap M_i} \pi_i(x_n),$$

which, because of Equation (3.2), can only happen if $a \cap M_i = b \cap M_i$ and $\varepsilon \upharpoonright M_i = \delta \upharpoonright M_i$. This allows us to cancel the terms corresponding to $M_i$, continuing with the process yields, after finitely many steps, that $a = b$ and $\varepsilon = \delta$.

□

The following theorem is the last piece needed for proving Theorem 3.27.

**Theorem 3.32.** *Let $G$ be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter. If*

$$\left\{x \in Q(G) \middle| \pi_{\rho(x)}(x) \in \left\{\frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{3}{8}\right\}\right\} \in p,$$

*then there exists a set $X \in p$ such that for every sequence $\vec{x}$ of elements of $\bigoplus_{n<\omega} \mathbb{T}$, if $\mathrm{FS}(\vec{x}) \subseteq X$ then $\vec{x}$ must satisfy 2-uniqueness of finite sums.*

*Proof.* If $p \in G^*$ is as described in the hypothesis, then there is an $i \in \{1, -1, 3, -3\}$ such that

$$Q_i = \left\{x \in Q(G) \middle| \pi_{\rho(x)}(x) = \frac{i}{8}\right\} \in p.$$

Let $\vec{x}$ be such that $p \ni \mathrm{FS}(\vec{x}) \subseteq Q_i$. For $j < \omega$ let $M_j = \{n < \omega \big| \rho(x_n) = j\}$.

**Claim 3.2.** *For each $j < \omega$, $|M_j| \leq 2$.*

*Proof of Claim.* Assume, by way of contradiction, that there are three distinct $n, m, k \in M_j$, and let $x = x_n + x_m + x_k$. For $l < j$, $\pi_l(x)$ must be an element of $\left\{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right\}$, because so are $\pi_l(x_n), \pi_l(x_m)$ and $\pi_l(x_k)$. On the other hand, $\pi_j(x_n) = \pi_j(x_m) = \pi_j(x_k) = \frac{i}{8}$, so $\rho(x) = j$ but $\pi_j(x) = \frac{3i}{8} \neq \frac{i}{8}$. □

Thus we can rearrange the sequence $\vec{x}$ in such a way that $n < m$ implies $\rho(x_n) \leq \rho(x_m)$, where the inequality is strict if $m > n + 1$. Let $M = \{\rho(x_n) \big| n < \omega\}$.

**Claim 3.3.** *Let $n < m < \omega$ and assume that $j = \rho(x_n) < \rho(x_m)$ (which may or may not hold if $m = n + 1$, but must hold if $m > n + 1$). Then $\pi_j(x_m) = 0$.*

*Proof of Claim.* Let $x = x_n + x_m$. Arguing as in the proof of Claim 3.2, we get that $\rho(x) = j$ and thus since $x \in Q_i$, $\pi_j(x_n) + \pi_j(x_m) = \pi_j(x) = \frac{i}{8}$. Now on the one hand we know that $\pi_j(x_m) \in \left\{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right\}$, while on the other hand $\pi_j(x_n) = \frac{i}{8}$. Hence the only possibility that does not lead to contradiction is that $\pi_j(x_m) = 0$. □

**Claim 3.4.** *For every $x \in \mathrm{FS}(\vec{x})$ there is a $j \in M$ such that $\pi_j(x) \neq 0$. Moreover for the least such $j$ we actually have that $\pi_j(x) \in \left\{\frac{i}{8}, \frac{2i}{8}\right\}$.*

*Proof of Claim.* For if $x = \sum_{n\in a} x_n$ and if $m = \min(a)$, then we can let $j = \rho(x_m) \in M$, so that for every $n \in a$ we have $\rho(x_n) \geq j$, with a strict inequality if $n > m+1$. Now, we have that

$$\pi_j(x) = \sum_{n\in a} \pi_j(x),$$

where, by Claim 3.3, each of the terms on the right-hand side of this expression are zero, except for $\pi_j(x_m) = \frac{1}{8}$ and possibly $\pi_j(x_{m+1})$ (which will appear on the summation only if $m+1 \in a$, and if so it will equal $\frac{1}{8}$ if $\rho(x_{m+1}) = \rho(x_m)$, and zero otherwise). Thus $\pi_j(x) \in \left\{\frac{i}{8}, \frac{2i}{8}\right\}$. In particular $\pi_j(x) \neq 0$, now in order to prove the "moreover" part, we will argue that for all $l < j$ such that $l \in M$, $\pi_l(x) = 0$. This is because if $l \in M$, then there is $k < \omega$ such that $\rho(x_k) = l$, and if $l < j$ then we must necessarily have $k < m$ because of the way we arranged our sequence $\vec{x}$. Hence, again by Claim 3.3 and since $m = \min(a)$, it will be the case that $\pi_l(x_n) = 0$ for all $n \in a$, and hence

$$\pi_l(x) = \sum_{n\in a} \pi_l(x_n) = 0,$$

therefore $j$ is actually the least $l \in M$ such that $\pi_l(x) \neq 0$ and we are done. □

The previous claim allows us to define $\tau : \mathrm{FS}(\vec{x}) \longmapsto M$ by $\tau(x) = \min\{j \in$

$M \middle| \pi_j(x) \neq 0\}$, and ensures that $\pi_{\tau(x)}(x) \in \left\{\frac{i}{8}, \frac{2i}{8}\right\}$. We can thus let

$$C_k = \left\{ x \in \mathrm{FS}(\vec{x}) \middle| \pi_{\tau(x)}(x) = \frac{ki}{8} \right\}$$

for $k \in \{1, 2\}$, and choose from among those the $k$ such that $C_k \in p$. We let $X = C_k$ and claim that $X$ is as in the conclusion of the theorem. In order to see this, let $\vec{y}$ be such that $\mathrm{FS}(\vec{y}) \subseteq C_k$.

Notice first that for distinct $n, m < \omega$ we must have $\tau(y_n) \neq \tau(y_m)$, for otherwise we would get, arguing in a similar way as in the proofs of Claims 3.2 and 3.3, that $\tau(y_n + y_m) = \tau(y_n) = \tau(y_m)$ and $\pi_{\tau(y_n+y_m)}(y_n + y_m) = \frac{2ki}{8} \neq \frac{ki}{8}$, a contradiction. Thus by rearranging $\vec{y}$ if necessary, we can assume that $n < m$ implies $\tau(y_n) < \tau(y_m)$.

Now an observation is in order. Consider $a \in [\omega]^{<\omega} \setminus \varnothing$ and $\varepsilon : a \longrightarrow \{1, 2\}$. Let $m = \min(a)$ and $j = \tau(y_m)$. Since $\tau$ is increasing on $\vec{y}$, $\pi_j(y_n) = 0$ for all $n \in a \setminus \{m\}$, while $\pi_j(y_m) = \frac{ki}{8}$. Thus

$$\pi_j \left( \sum_{n \in a} \varepsilon(n) y_n \right) = \varepsilon(m) \frac{ki}{8} \neq 0.$$

From this we can conclude that $\vec{y}$ satisfies 2-uniqueness of finite sums. Assume that $a, b \in [\omega]^{<\omega}$ and $\varepsilon : a \longrightarrow \{1, 2\}, \delta : b \longrightarrow \{1, 2\}$ are such that

$$\sum_{n \in a} \varepsilon(n) x_n = \sum_{n \in b} \delta(n) x_n. \tag{3.3}$$

We will proceed by induction on $\min\{|a|, |b|\}$. If $a = b = \varnothing$ we are done. Otherwise let $m = \min(a \cup b)$. Assume without loss of generality that $m \in a$, so that $m =$

min$(a)$. Let $j = \tau(y_m)$. Then by the previous observation, the value of each side of (3.3) under $\pi_j$ is nonzero, while $\pi_j(y_n) = 0$ for all $n > m$, thus by looking at the right-hand side of (3.3) we conclude that we must have $m \in b$ as well. Then it is also the case that $\min(b) = m$. Now again, by the observation from last paragraph we get that the value of each side of (3.3) under the function $\pi_j$ must equal, at the same time, $\varepsilon(m)\frac{ki}{8}$ and $\delta(m)\frac{ki}{8}$. This can only happen if $\varepsilon(m) = \delta(m)$, therefore we can cancel the term $\varepsilon(m)y_m$ from both sides of (3.3) and get

$$\sum_{n \in a \setminus \{m\}} \varepsilon(n)x_n = \sum_{n \in b \setminus \{m\}} \delta(n)x_n,$$

now we can apply the inductive hypothesis and conclude that $a \setminus \{m\} = b \setminus \{m\}$ and $\varepsilon \restriction (a \setminus \{m\}) = \delta \restriction (b \setminus \{m\})$. Since $m$ is an element of both $a$ and $b$, with $\varepsilon(m) = \delta(m)$, we have proved that $a = b$ and $\varepsilon = \delta$, and we are done. $\square$

*Proof of Theorem 3.27.* Let $G$ be an abelian group, and $p \in G^*$ be a strongly summable ultrafilter such that $\{x \in G \big| o(x) = 2\} \notin p$. Since $p$ is nonprincipal and the only $x \in G$ with $o(x) = 1$ is 0, we have that $B = \{x \in G \big| o(x) > 2\} \in p$. Hence, there are two possibilities: If $C = \{x \in G \big| o(x) = 4\} \in p$, then we can pick a sequence $\vec{x}$ such that $p \ni \mathrm{FS}(\vec{x}) \subseteq C$, so by Lemma 3.29 this sequence must satisfy 2-uniqueness of finite sums and we are done. Otherwise, if $C \notin p$ then we have that (since $C \cup B \cup \{0\} = Q(G)$)

$$G \setminus Q(G) = \{x \in G \big| o(x) \notin \{1, 2, 4\}\} \in p.$$

Now $G \setminus Q(G) = Q_0 \cup Q_1$, where

$$Q_0 = \left\{ x \in G \setminus Q(G) \middle| \pi_{\rho(x)}(x) \notin \left\{ \frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{3}{8} \right\} \right\},$$

and

$$Q_1 = \left\{ x \in G \setminus Q(G) \middle| \pi_{\rho(x)}(x) \in \left\{ \frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{3}{8} \right\} \right\},$$

so pick $i \in 2$ such that $Q_i \in p$. If $i = 0$ apply Theorem 3.31 and if $i = 1$ apply Theorem 3.32, in either case, there is an $X \in p$ such that whenever $\vec{x}$ is such that $\mathrm{FS}(\vec{x}) \subseteq X$, then $\vec{x}$ must satisfy 2-uniqueness of finite sums. By strong summability of $p$ there is such a sequence $\vec{x}$ which additionally satisfies $\mathrm{FS}(\vec{x}) \in p$, and we are done. □

**Corollary 3.33** ([20], Question 4.12)**.** *Let $p$ be a nonprincipal strongly summable ultrafilter on an abelian group $G$. Then $p$ is sparse.*

*Proof.* Let $G$ be any abelian group, and let $p \in G^*$ be a strongly summable ultrafilter. Let

$$B = \{x \in G \big| o(x) \leq 2\}.$$

Then $B$ is a subgroup of $G$. If $B \in p$ then since $p$ is nonprincipal, $B$ must be infinite; and since $G$ is countable, $B$ must be isomorphic to the (unique up to isomorphism) countably infinite Boolean group. Consider the restricted ultrafilter $q = p \upharpoonright B = p \cap \mathfrak{P}(B)$. Then $q$ is also strongly summable, so $q$ is a nonprincipal strongly summable ultrafilter on the Boolean group and therefore by Theorem 3.17 it is sparse. It is easy to see that this implies that $p$ is sparse as well. Thus the only case that remains to be proved is when $B \notin p$, but this is handled by Theorem 3.27 together with Corollary 3.23, and we are done. □

**Corollary 3.34** ([20], Question 4.11)**.** *Let $p$ be a nonprincipal strongly summable ultrafilter on an abelian group $G$. Then $p$ has the trivial sums property.*

*Proof.* Let $G$ be any abelian group, and let $p \in G^*$ be a strongly summable ultrafilter. If $p$ does not contain the subgroup $B = \{x \in G \big| o(x) \leq 2\}$, then we just need to apply Theorems 3.27 and 3.8. So assume that $B \in p$ and let $q, r \in \beta G$ be such that $q + r = p$. Then we have that

$$\{x \in G \big| B - x \in r\} \in q,$$

in particular this set is nonempty and so we can pick an $x \in G$ such that $B - x \in r$, or equivalently $B \in r + x$. Since $x \in G$ (hence it commutes with all ultrafilters), the equation $(q - x) + (r + x) = p$ holds, thus

$$A = \{y \in G \big| B - y \in r + x\} \in q - x.$$

Notice that $A \subseteq B$, because if $y \in G$ is such that $B - y \in r + x$ then $B \cap (B - y) \in r + x$, in particular the latter set is nonempty and so there are $z, w \in B$ such that $z = w - y$ which means that $y = w - z \in B$. Therefore $B \in q - x$, so we can define

$u = (q - x) \restriction B$ and $v = (r + x) \restriction B$. We then get that $u, v \in \beta B$ and $p \restriction B \in B^*$ is a strongly summable ultrafilter such that $u + v = p \restriction B$. By Theorem 3.18, we conclude that $u, v \in B + p \restriction B$, which is easily seen to imply that $q - x, r + x \in B + p$, and therefore, since $x \in G$, we conclude that $q, r \in G + p$ and we are done. $\square$

# 4 Finer Existence Results

We turn our eye again to existence questions. Because of the results from the previous chapter, we do not suffer a terrible loss of generality if we focus exclusively on the Boolean group, so the first thing we do is show some properties of strongly summable ultrafilters on this group. We then establish that the statement "there exists a strongly summable ultrafilter on the Boolean group" is consistent with $\text{cov}(\mathcal{M}) < \mathfrak{c}$ together with each of the assumptions $\mathfrak{d} = \mathfrak{c} = \omega_2$ and $\mathfrak{d} < \mathfrak{c}$ (recall that this statement was already known to follow from just $\text{cov}(\mathcal{M}) < \mathfrak{c}$, although its negation is also known to be consistent with $\mathsf{ZFC}$). To close this dissertation, we prove that, assuming $\text{cov}(\mathcal{M}) = \mathfrak{c}$, there exists on the Boolean group a strongly summable ultrafilter that is not additively isomorphic to any union ultrafilter.

## 4.1 Strongly Summable Ultrafilters on the Boolean group

In this section we will focus on the Boolean group, and on the properties of strongly summable ultrafilters on this group. Most of these results had already been proved for the case of union ultrafilters, which are particular cases of strongly summable ultrafilters on $\mathbb{B}$. The slightly more general theorems that we present here are proved in much the same way as in the particular cases for union ultrafilters, with help from the following lemma which will reveal some of the internal structure of sets $\mathrm{F}\triangle(X)$. We will be using the functions $\max : \mathbb{B} \longrightarrow \omega$ and $\min : \mathbb{B} \longrightarrow \omega$ (recall that the group $\mathbb{B}$ has $[\omega]^{<\omega}$ as its underlying set).

**Lemma 4.1.** *Let $X \subseteq \mathbb{B}$ be a linearly independent set. It is possible to find a linearly independent $Y$ such that $\mathrm{F}\triangle(Y) = \mathrm{F}\triangle(X)$ and $\max \upharpoonright Y$, $\min \upharpoonright Y$ are injective (in particular, $\max[\mathrm{F}\triangle(Y)] = \max[Y]$ and $\min[\mathrm{F}\triangle(Y)] = \min[Y]$).*

*Proof.* Let $X \in [\mathbb{B}]^\omega$ be linearly independent. Fix an increasing enumeration $\{n_k \big| k < \omega\}$ of $\max[\mathrm{F}\triangle(X)]$.

**Claim 4.1.** *For each $k < \omega$, the set $X_k = \{z \in \mathrm{F}\triangle(X) \big| \max(z) = n_k\}$ has exactly $2^k$ elements.*

*Proof.* We proceed by induction on $k$. There cannot be two distinct $y, z \in X_0$, for otherwise $\varnothing \neq y \triangle z \in \mathrm{F}\triangle(X)$ and $\max(y \triangle z) < n_0$, contradicting the definition of $n_0$. Now assume that we have already proven that $|X_j| = 2^j$ for all $j < k$. Let $\{y_i \big| i < \sum_{j<k} 2^j = 2^k - 1\}$ be an enumeration of $\bigcup\limits_{j<k} X_j$. Pick any $x \in \mathrm{F}\triangle(X)$ such that $\max(x) = n_k$. Then the set $\{x\} \cup \{x \triangle y_i \big| i < 2^k - 1\}$ consists of $2^k$ distinct elements of $\mathrm{F}\triangle(X)$, all of them with maximum $n_k$. Moreover, if $z \in \mathrm{F}\triangle(X)$ is such that $\max(z) = n_k$, then either $z = x$ or $\max(x \triangle z) < n_k$, thus $x \triangle z = y_i$ for some $i < 2^k - 1$. Then $z = x \triangle y_i$ lies in the aforementioned set. This proves that $X_k = \{x\} \cup \{x \triangle y_i \big| i < 2^k - 1\}$ and we are done. □

The previous argument shows even more, namely it shows that however we choose an element $x_n \in X_n$ for each $n < \omega$, the resulting set $Y = \{x_n \big| n < \omega\}$ is linearly independent and $\mathrm{F}\triangle(Y) = \mathrm{F}\triangle(X)$. This will allow us to recursively set up a choice of an $x_n \in X_n$, for each $n < \omega$, such that the $x_n$ have pairwise disjoint minima. We first let $x_0$ be the unique element of $X_0$. We now assume that we have chosen $x_0, \ldots, x_k$ with $x_i \in X_i$ and pairwise disjoint minima (let $m_i = \min(x_i)$), and explain how to choose $x_{k+1}$. We start by choosing any $y \in X_{k+1}$. We let $a = y \cap \{m_i | i \leq k\}$ and define

$$x_{k+1} = y \triangle \left( \bigtriangleup_{i \in a} x_i \right).$$

It is readily checked that $x_{k+1} \in X_{k+1}$ and that for all $i \leq k$, $m_i \notin x_{k+1}$, in particular $m_{k+1} = \min(x_{k+1}) \neq m_i$ for any $i \leq k$. In the end we just collect what we chose into the family $Y = \{m_k \big| k < \omega\}$, and we are done. □

The previous lemma will allow us to extend some results of Blass, for union ultrafilters, to the more general context of strongly summable ultrafilters on the Boolean group. Recall that an ultrafilter $p \in \omega^*$ is said to be a **P-point** if whenever $A_n \in p$ for $n < \omega$, it is possible to find a pseudointersection $A$ for the $A_n$ (this is,

$A \subseteq \omega$ is infinite and $(\forall n < \omega)(A \subseteq^* A_n))$ such that $A \in p$. It is not terribly hard to see [1] that, equivalently, $p \in \omega^*$ is a P-point if and only if for every function $f : \omega \longrightarrow \omega$ there exists an $A \in p$ such that $f \upharpoonright A$ is either constant or finite-to-one. The following theorem was originally proved by Blass and Hindman [6, Theorem 2] for union ultrafilters only.

**Theorem 4.2.** *If $p \in \mathbb{B}^*$ is a strongly summable ultrafilter, then* $\min(p)$ *is a P-point.*

*Proof.* Let $f : \omega \longrightarrow \omega$, and we will find an element of $\min(p)$ on which $f$ is either constant or finite-to-one. For $x \in \mathbb{B} \setminus \{\varnothing\}$, we let $\varphi(x)$ denote the number of "consecutive pairs" $i < j$ (this means that $i, j \in x$ but for every $i < k < j$, $k \notin x$) such that $f(j) \leq i$. Notice that whenever $\max(x) < \min(y)$, then $\varphi(x \triangle y) = \varphi(x) + \varphi(y) + \Delta(x, y)$ where we define $\Delta(x, y)$ to equal 1 if $f(\min(y)) \leq \max(x)$ and 0 otherwise. For $i \in 2$, we define

$$A_i = \{x \in \mathbb{B} \big| \varphi(x) \equiv i \mod 2\}.$$

Then $A_0 \cup A_1$ defines a partition of $\mathbb{B}$, so we can choose $i \in 2$ such that $A_i \in p$, and by strong summability we can also pick a linearly independent $X$ such that $p \ni \mathrm{F}\triangle(X) \subseteq A_i$. Moreover by Lemma 4.1, we can assume that $\max \upharpoonright X$ and $\min \upharpoonright X$ are both injective.

Now there are two cases. If $i = 1$, fixing any $x \in X$ we have that, whenever

$y \in \mathrm{F}\triangle(X)$ is such that $\max(x) < \min(y)$, we must have that $f(\min(y)) \leq \max(x)$ because otherwise it would be impossible that all three of $\varphi(x), \varphi(y), \varphi(x \triangle y)$ are simultaneously odd. But since $\min \upharpoonright X$ is injective, by dropping finitely many elements from $X$ we can get a $Y$ such that $(\forall y \in \mathrm{F}\triangle(Y))(\max(x) < \min(y))$, and certainly it will be the case that $\mathrm{F}\triangle(Y) \in p$. Hence for every $y \in \mathrm{F}\triangle(Y)$, we have that $f(\min(y)) \leq \max(x)$, so since $\min[Y] = \min[\mathrm{F}\triangle(Y)]$ we get that $\min[Y] \in \min(p)$ and $f \upharpoonright \min[Y]$ has finite range (it can only take values $\leq \max(x)$). Thus one of the finitely many fibers belongs to $\min(p)$ and so we get a set in $\min(p)$ where $f$ is constant.

The second case is when $i = 0$. Then, in order to preserve the parity, given any $x, y \in \mathrm{F}\triangle(X)$ such that $\max(x) < \min(y)$, we must have that $\max(x) < f(\min(y))$ (so that all three of $\varphi(x), \varphi(y), \varphi(x \triangle y)$ are even). Thus given any $k < \omega$, we can pick an $x \in X$ such that $\max(x) \geq k$ and then we will have that, except for the (at most) finitely many $y \in X$ such that $\min(y) < \max(x)$, every other $y \in X$ has to satisfy that $f(\min(y)) > \max(x) \geq k$. Hence there are only finitely many elements from $\min[X]$ whose image under $f$ is $k$; so since $\min[X] = \min[\mathrm{F}\triangle(X)] \in \min(p)$ we get our set in $p$ where $f$ is finite-to-one. $\square$

**Corollary 4.3.** *It is not possible to prove the existence of strongly summable ultrafilters on any abelian group in* $\mathsf{ZFC}$.

*Proof.* Since any strongly summable ultrafilter on an abelian group is additively isomorphic to some union ultrafilter, which in turn gives rise (via the mapping min) to a P-point. However, by a result of Shelah's (see [1, Theorem 4.4.7] or [35] for somewhat understandable proofs), there are models of ZFC in which P-point ultrafilters do not exist. □

Recall that a nonprincipal ultrafilter $p$ on $\omega$ is called **rapid** if the collection of all enumerating functions from elements of $p$ form a dominating family in $({}^{\omega}\omega, \leq)$. If we fix beforehand an unbounded function $g : \omega \longrightarrow \omega$, then rapidity of $p$ is equivalent, by [8, Lemma 2.2.5], to asking that for every finite-to-one function $f : \omega \longrightarrow \omega$, there exists an $A \in p$ such that $(\forall n < \omega)(|A \cap f^{-1}[\{n\}]| \leq g(n))$ (rephrasing in terms of partitions, we would say that $p$ is rapid iff for every partition $\{F_n \big| n < \omega\}$ of $\omega$ into finite sets, there exists an $A \in p$ such that $(\forall n < \omega)(|A \cap F_n| < g(n))$). Unlike the original definition, the latter characterization does not rely on the way we choose to order $\omega$, hence it can be taken as definition being of rapid for an ultrafilter $p$ on any countable set $X$. For this reason, it is possible to ask the question of whether an ultrafilter on a countable set is rapid. It is also worth noticing that if $Y \subseteq X$ are both countable and $p$ is an ultrafilter on $X$ with $Y \in p$, then $p$ is rapid if and only if so is $p \restriction Y$.

We will now talk about the image of $p$ under max. In order to do so, we will need the following lemma.

**Lemma 4.4.** *Let $n, i < \omega$. If we have $2n$ elements $x_0, \ldots, x_{2n-1} \in \mathbb{B}$ with pairwise* *distinct maxima, and such that* $\min(x_k) \geq i$ *for all* $k < 2n$*, then it is possible to find* $n$ *elements* $y_0, \dots, y_{n-1} \in \mathrm{F}\triangle(\langle x_k \big| k < 2n\rangle)$ *with pairwise distinct maxima such that* $\min(y_k) \geq i+1$ *for all* $k < n$*.*

*Proof.* Let $m$ be the amount of those $k < 2n$ such that $\min(x_k) \geq i+1$. If $m \geq n$ we are done, otherwise there are $2n - m > 2(n-m)$ elements $x_{k_0}, \dots, x_{k_{2n-m-1}}$ such that $\min(x_{k_j}) = i$ for all $j < 2n - m$. Let $y_j = x_{k_{2j}} \triangle x_{k_{2j+1}}$ for $j < n - m$. Since the $x_k$ have pairwise distinct maxima, it is easy to see that they are linearly independent, so the $y_j$ thus defined are nonzero and elements of $\mathrm{F}\triangle(\langle x_k \big| k < 2n\rangle)$; also clearly $\min(y_j) \geq i+1$. Now for $n - m \leq j < n$, just let each $y_j$ be one of those $x_k$ with $\min(x_k) \geq i+1$. We are done once we observe that defining the $y_j$ ($j < n$) this way produces them with pairwise distinct maxima. □

The previous lemma allows us to state the following result, which is a generalization to all strongly summable ultrafilters on $\mathbb{B}$ of a result that Blass and Hindman [6, Theorem 2] (partially) and Matet [31] proved for union ultrafilters.

**Theorem 4.5.** *If* $p \in \mathbb{B}^*$ *is strongly summable, then* $\max(p)$ *is a rapid P-point.*

*Proof.* We want to show that $\max(p)$ is at the same time rapid and a P-point, so it suffices to show that for every function $f : \omega \longrightarrow \omega$, one can find an element $A \in p$ such that either $f \upharpoonright A$ is constant, or for every $n < \omega$, the fibre $f^{-1}[\{n\}] \cap \max[A]$ has cardinality less than $2^n$ (this goes in two steps, as P-pointness allows us to pick $A \in p$ such that $f \restriction A$ is either constant or finite-to-one, and if the latter holds then we use rapidity to further shrink $A$ so it satisfies the relevant condition on the size of the fibres). First let $i \in 2$ be such that $A_i \in p$, where $A_0 = \{x \in \mathbb{B} \big| f(\max(x)) \leq \min(x)\}$ and $A_1 = \mathbb{B} \setminus A_0$. If $i = 0$, pick $X$ such that $p \ni \mathrm{F}\triangle(X) \subseteq A_0$, and then pick any $x \in X$. We certainly have that $B = \{z \in \mathrm{F}\triangle(X) \big| \min(z) > \max(x)\} \in p$. Notice that, for every $z \in B$, $x \cup z = z \triangle x \in \mathrm{F}\triangle(X) \subseteq A_0$, from where we conclude that $f(\max(z)) = f(\max(z \cup x)) \leq \min(z \cup x) = \min(x)$, so that $f \restriction \max[B]$ is bounded by $\min(x)$, which certainly implies that $f$ is constant at a set in $\max(p)$. Hence we are left with the case where $i \neq 0$, and $A_1 \in p$. Now find a linearly independent $X$ with $p \ni \mathrm{F}\triangle(X) \subseteq A_1$. Assume that for some $n$, the fibre $f^{-1}[\{n\}] \cap \max[\mathrm{F}\triangle(X)]$ has cardinality at least $2^n$. This means that it is possible to find $2^n$ elements $x_0, \dots, x_{2^n-1} \in \mathrm{F}\triangle(X)$, with pairwise distinct maxima, such that $f(\max(x_k)) = n$ for all $k < 2^n$. Use Lemma 4.4 to get $2^{n-1}$ elements $y_0, \dots, y_{2^{n-1}-1} \in \mathrm{F}\triangle(\langle x_k \big| k < 2^n \rangle) \subseteq \mathrm{F}\triangle(X)$, with pairwise distinct maxima, and such that $\min(y_k) \geq 1$ for $k < 2^{n-1}$. Notice that, since the $y_k$ are linear combinations of the $x_k$, which in turn have pairwise distinct maxima, we have that $\{\max(y_k) \big| k < 2^{n-1}\} \subseteq \{\max(x_k) \big| k < 2^n\}$, so $f(\max(y_k)) = n$ for all $k < 2^{n-1}$. Use the Lemma again to get now $2^{n-2}$ elements $z_0, \dots, z_{2^{n-2}-1} \in \mathrm{F}\triangle(\langle y_k \big| k < 2^{n-1} \rangle) \subseteq \mathrm{F}\triangle(X)$ with pairwise distinct maxima, such that $\min(z_k) \geq 2$ and $f(\max(z_k)) = n$ for all $k < 2^{n-2}$. Repeat this process, using iteratively Lemma 4.4, and at the $n$-th iteration we will have gotten one ($1 = 2^{n-n}$) element $w \in \mathrm{F}\triangle(X) \subseteq A_1$ such that $\min(w) \geq n = f(\max(w))$, a contradiction. □

**Corollary 4.6.** *There are no strongly summable ultrafilters on any abelian group in Laver's, Mathias's, Miller's or Solovay's (random) model.*

*Proof.* This follows from the fact that any strongly summable ultrafilter on some abelian group is additively isomorphic to one on $\mathbb{B}$, which in turn has an image under max that is a rapid P-point. However, there are no rapid ultrafilters in Laver's, Mathias's or Miller's models [32]. On the other hand, while there are both rapid ultrafilters and P-points in Solovay's model, there are no ultrafilters that are both rapid and P-points simultaneously [25]. □

We will now proceed to prove that, if $p \in \mathbb{B}^*$ is strongly summable, then $p$ itself is a rapid ultrafilter. Restricted to union ultrafilters, this result is due to Krautzberger [24].

**Theorem 4.7.** *If $p \in \mathbb{B}^*$ is strongly summable, then it is rapid.*

*Proof.* Let $\{F_n \big| n < \omega\}$ be a partition of $\mathbb{B} = [\omega]^{<\omega}$ into finite sets, we will show that there is an $A \in p$ such that $(\forall n < \omega)(|A \cap F_n| < 2^{n+1})$. Let $f : \omega \longrightarrow \omega$ be given by $f(n) = \max(\bigcup F_n) = \max(\max[F_n])$. Since $\max(p)$ is rapid, we can find $B \in \max(p)$ whose enumerating function dominates $f$. Since $p$ is strongly summable, and because of Lemma 4.1, we can assume without loss of generality that $B = \max[X]$ for some linearly independent $X \subseteq \mathbb{B}$ such that $\mathrm{F}\triangle(X) \in p$, now we claim that letting $A = \mathrm{F}\triangle(X)$ works. In order to see that, let $n < \omega$, and let $\{m_k \big| k < \omega\}$ be an increasing enumeration of $\max[\mathrm{F}\triangle(X)] = \max[X]$. Notice that by definition of $f$, we must have that $\max(z) \leq f(n)$ if $z \in F_n$; now (the enumerating function of) $B$ dominates $f$ so $m_n \geq f(n)$. Therefore, if $z \in \mathrm{F}\triangle(X) \cap F_n$ then $\max(z) = m_k$ for some $k \leq n$. Again by Lemma 4.1, we have that $|\{z \in \mathrm{F}\triangle(X) \big| \max(z) = m_k \text{ for some } k \leq n\}| = \sum_{k \leq n} 2^k = 2^{n+1} - 1$, from where we can conclude that $|\,\mathrm{F}\triangle(X) \cap F_n| < 2^{n+1}$. □

Now we turn our attention to the issue of near-coherence. Recall that two ultrafilters $p, q$ on $\omega$ are said to be **near-coherent** if there exists a finite-to-one function $f : \omega \longrightarrow \omega$ such that $f(p) = f(q)$. We will state without proof the following useful characterization of near-coherence of ultrafilters, due to Eisworth [9, Propositions 2.1 and 2.2].

**Lemma 4.8** (Eisworth)**.** *Given two ultrafilters $p$ and $q$ on $\omega$, the following three conditions are equivalent.*

*(i) $p$ and $q$ are not near coherent,*

*(ii) For every partition of $\omega$ into intervals $I_n$, there exist two sets $X \in p$ and $Y \in q$ such that, for every two $n, m < \omega$, if $X \cap I_n \neq \varnothing$ and $Y \cap I_m \neq \varnothing$ then $n \neq m$.*

*(iii)* *For every partition of $\omega$ into intervals $I_n$, there exist two sets $X \in p$ and $Y \in q$ such that, for every two $n, m < \omega$, if $X \cap I_n \neq \varnothing$ and $Y \cap I_m \neq \varnothing$ then $n \neq m$ and moreover there exists a $k$ between $n$ and $m$ such that $X \cap I_k = \varnothing = Y \cap I_k$ (this is, there is a "buffer" interval $I_k$ which meets neither of $X, Y$).*

Next, we show that for every strongly summable ultrafilter $p \in \mathbb{B}^*$, the ultrafilters $\max(p)$ and $\min(p)$ are not near-coherent. The proof is a simple modification of the proof for the particular case of union ultrafilters which is due to Blass [4, Theorem 38].

**Theorem 4.9.** *If $p \in \mathbb{B}^*$ is strongly summable, then* $\max(p)$ *and* $\min(p)$ *are not near-coherent.*

*Proof.* Assuming the opposite, we would be able to find a partition of $\omega$ into intervals, $\{I_n \big| n < \omega\}$, such that for every set $A \in p$, there are infinitely many $n < \omega$ with $\max[A] \cap I_n \neq \varnothing \neq \min[A] \cap I_n$. Given $x \in \mathbb{B} \setminus \{\varnothing\}$, denote the number $|\{n < \omega \big| I_n \cap x \neq \varnothing\}|$ by $\varphi(x)$. Let $\mathbb{B} \setminus \{\varnothing\} = A_0 \cup A_1 \cup A_2$, where

$$A_i = \{x \in \mathbb{B} \setminus \{\varnothing\} \big| \varphi(x) \equiv i \mod 3\}.$$

If $i \in 3$ is such that $A_i \in p$, then it cannot be the case that $i \neq 0$, for otherwise, by choosing an $X$ with $p \ni \mathrm{F}\triangle(X) \subseteq A_i$, and finding $x, y \in \mathrm{F}\triangle(X)$ such that $\min(y) > \max\left(\bigcup_{I_n \cap x \neq \varnothing} I_n\right)$, we would have that $\varphi(x \triangle y) = \varphi(x) + \varphi(y) \equiv 2i \not\equiv i$

mod 3. Hence $A_0 \in p$, so let $X$ be a linearly independent set such that $p \ni \mathrm{F}\triangle(X) \subseteq A_0$. Then by assumption, we can find an $n < \omega$ and $x, y \in \mathrm{F}\triangle(X)$ such that $\max(x) \in I_n$ and $\min(y) \in I_n$. Then $x \triangle y \in \mathrm{F}\triangle(X)$, and $\varphi(x \triangle y) = \varphi(x) + \varphi(y) - k$, where $k$ equals 2 if $x \cap I_n = y \cap I_n$, and 1 otherwise. But $\varphi(x) \equiv \varphi(y) \equiv 0 \mod 3$, so in any case, $\varphi(x \triangle y) \equiv -k \not\equiv 0 \mod 3$, thus contradicting that $\mathrm{F}\triangle(X) \subseteq A_0$. Therefore $\max(p)$ and $\min(p)$ are not near-coherent. □

This provides another proof of the part of Corollary 4.6 that refers to Miller's model, as this is a model that satisfies $\mathsf{NCF}$, meaning that every two ultrafilters on this model are near-coherent. It also allows us to give an alternative proof of Theorem 3.21 for the case of ordered union ultrafilters.

**Lemma 4.10.** *If $p \in \mathbb{B}^*$ is an ordered union ultrafilter, then for every $A \in p$ it is possible to find an ordered family $X = \{x_n \big| n < \omega\}$ (where $\max(x_n) < \min(x_{n+1})$ for every $n < \omega$) such that for some coinfinite subfamily $Y = \{x_{n_k} \big| k < \omega\}$ (i.e. $\{n_k \big| k < \omega\}$ is coinfinite) we have that $\mathrm{F}\triangle(Y) \in p$.*

*Proof.* Let $p \in \mathbb{B}^*$ be an ordered union ultrafilter and let $A \in p$. Then we can grab an ordered family $X = \{x_n \big| n < \omega\} \subseteq B$ (with $\max(x_n) < \min(x_{n+1})$) such that $p \ni \mathrm{F}\triangle(X) \subseteq A$, which very naturally defines a partition into intervals. More precisely, it is easy to get a partition into intervals $\omega = \bigcup_{n<\omega} I_n$ such that $(\forall n < \omega)(x_n \subseteq I_n)$. Therefore, Theorem 4.9 makes it possible to find an ordered family $Y = \{y_n \big| n < \omega\}$ such that $\mathrm{F}\triangle(Y) \in p$, $\mathrm{F}\triangle(Y) \subseteq \mathrm{F}\triangle(X)$ and such that, for every $n < \omega$, if $k_n$ is such that $\max(y_n) \in I_{k_n}$ then $\min(y_{n+1}) \notin I_{k_n+1}$. Hence if we define $Z = Y \cup \{x_{k_n} \big| n < \omega\}$, we will have that $Z$ is an ordered family such that $\mathrm{F}\triangle(Z) \subseteq \mathrm{F}\triangle(X) \subseteq A$ and we can still drop infinitely many elements of $Z$ (namely all of the $x_{k_n+1}$) and get $Y$ with $\mathrm{F}\triangle(Y) \in p$. $\square$

We will now address the "classical" models of Set-Theory, by which at this moment we mean the models that appear on the table at the end of Blass's article on Cardinal Invariants in the Handbook of Set Theory [5]. Thus the models we consider are: $\mathsf{MA}$, Cohen, Random, Sacks, Hechler, Laver, Mathias and Miller. Notice first of all that, since $\mathsf{MA}$, Cohen and Hechler satisfy $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, we settle at once the question of whether these models have strongly summable ultrafilters in the affirmative. In the case of Laver's, Mathias's, Miller's and Solovay's (random) model, we already pointed out, in Corollary 4.6, that for no abelian group $G$ are there any nonprincipal strongly summable ultrafilters on $G$. Thus, it seems that the question of whether strongly summable ultrafilters exist in these models can be settled relatively easily except for Sacks model. The author (with the help from several people, including his supervisor Juris Steprāns) spent some time trying to settle this question, and has so far been unsuccessful in this enterprise. Thus, we would like to close this section by stating that question.

**Question 4.11.** *Are there strongly summable ultrafilters (on some abelian group, so without loss of generality in $\mathbb{B}$) in Sacks's model?*

## 4.2 Stability and Games

In this section we introduce a notion which will be important for some results of Section 4.3. It was originally introduced by Blass [3, p. 94] in the context of union ultrafilters but we will treat it in full generality.

**Definition 4.12.** A strongly summable ultrafilter $p \in \mathbb{B}^*$ is said to be **stable** if whenever $\{A_n \big| n < \omega\} \subseteq p$, there exists a linearly independent $X$ such that $\mathrm{F}\triangle(X) \in p$ and for all $n < \omega$, there exists a finite $F \subseteq X$ with $\mathrm{F}\triangle(X \setminus F) \subseteq A_n$.

It is easy to see that the definition for a stable strongly summable ultrafilter is equivalent to the following: a strongly summable $p \in \mathbb{B}^*$ is stable if and only if, for every countably many linearly independent sets $\{X_n \big| n < \omega\}$ such that $(\forall n < \omega)(\mathrm{F}\triangle(X_n) \in p)$, there exists a linearly independent $X$ such that $\mathrm{F}\triangle(X) \in p$ and $(\forall n < \omega)(X \subseteq^* \mathrm{F}\triangle(X_n))$. We say that $X$ is a **common pseudocondensation** for the $X_n$.

A stable ordered union ultrafilter (this is, an ordered union ultrafilter which happens to also be stable) is what Matet [31] calls a *Milliken-Taylor ultrafilter*. The objective of this section is to develop a characterization of stable ordered union ultrafilters in terms of games, which will be used heavily in Section 4.3. We warn the reader that our game is different from the one used by Matet [31, p. 548], and that this characterization is the result of a joint work with David Chodounský and Peter Krautzberger (still unpublished).

The first tool that we will need for our characterization is a result that was first proved by Blass [3, Theorem 4.2] for ordered union ultrafilters, and later on generalized by Krautzberger [23, Theorem 4.2] for union ultrafilters in general. We note here that, although Krautzberger's argument actually works for every strongly summable ultrafilter on $\mathbb{B}$, we were able to get a somewhat simpler (or so we think) argument.

**Definition 4.13.** We say that an ultrafilter $p \in \mathbb{B}^*$ has the **Ramsey property for pairs** if whenever the set $\mathbb{B}^2_< = \{(x, y) \in \mathbb{B}^2 \big| \max(x) < \min(y)\}$ is coloured into finitely many colours, there exists an $A \in p$ such that $A^2_< = \{(x, y) \in A^2 \big| \max(x) < \min(y)\}$ is monochromatic.

**Theorem 4.14.** *A strongly summable ultrafilter $p \in \mathbb{B}^*$ is stable if and only if it has the Ramsey property for pairs.*

*Proof.* Assume first that $p$ is strongly summable with the Ramsey property for pairs, and let $\{A_n \big| n < \omega\} \subseteq p$. Without loss of generality the $A_n$ are decreasing and $\bigcap_{n<\omega} A_n = \varnothing$, so it makes sense to define, for $x \in A_0$, the number $f(n)$ to be the unique $k$ such that $x \in A_k \setminus A_{k+1}$. We colour a pair $(x, y) \in {A_0}^2_<$ on colour white if an only if $f(x) = f(y)$ (and in colour black otherwise) and grab a linearly independent family $X$ such that $\mathrm{F}\triangle(X) \in p$, $\mathrm{F}\triangle(X) \subseteq A_0$, and $\mathrm{F}\triangle(X)^2_<$ is monochromatic. We claim that $X$ is a common pseudocondensation. We will prove by induction on $n < \omega$ that there is a cofinite subset $Y \subseteq X$ such that $\mathrm{F}\triangle(Y) \subseteq A_n$. Certainly $\mathrm{F}\triangle(X) \subseteq A_0$. Assume that we have the claim proved for $n$ and let $Y \subseteq X$ be a cofinite subset such that $\mathrm{F}\triangle(Y) \subseteq A_n$. If we actually have $\mathrm{F}\triangle(Y) \subseteq A_{n+1}$ we are done, so we may assume that there is a $y \in \mathrm{F}\triangle(Y) \setminus A_{n+1}$. This means that $f(y) = n$. Now, because of Lemma 4.1, the set $Z = \{z \in Y \big| \max(y) < \min(z)\}$ is cofinite in $Y$ (hence also in $X$), and all of the pairs $(y, z)$, for $z \in \mathrm{F}\triangle(Z)$, receive the same colour. If that colour was white, it would mean that $f(y) = f(z) = n$ for all $z \in \mathrm{F}\triangle(Z)$, hence $\mathrm{F}\triangle(Y) \subseteq A_n \setminus A_{n+1} \notin p$, a contradiction. Therefore the colour must be black, so $f(y) \neq f(z)$ for all $z \in \mathrm{F}\triangle(Z)$, and since we know that $\mathrm{F}\triangle(Z) \subseteq A_n$, we can conclude that $f(y) = n < f(z)$ for all $z \in \mathrm{F}\triangle(Z)$. This means that $\mathrm{F}\triangle(Z) \subseteq A_{n+1}$, and we are done.

Conversely, assume that $p$ is a stable strongly summable ultrafilter, and assume that we have coloured all ordered pairs from $\mathbb{B}^2_<$ into two colours (say, black or white). For each $x \in \mathbb{B}$ we can partition the set $\{y \in \mathbb{B} \big| \min(y) > \max(x)\} \in p$ depending on whether the pair $(x, y)$ is black or white. We let $p$ choose an element of the partition (so, $p$ chooses one of the two colours for $x$). We have thus partitioned

$\mathbb{B}$ into two cells, according to the colour that $p$ chose for each $x \in \mathbb{B}$ and now we let $p$ choose one of those cells. Without loss of generality we assume that $p$ chooses the colour white. This is, there exists an $A \in p$ such that for all $x \in A$, the set

$$A_x = \{y \in A \big| \min(y) > \max(x) \text{ and } (x, y) \text{ is white}\} \in p$$

For each $n < \omega$ we pick a linearly independent family $X_n$ such that

$$p \ni \mathrm{F}\triangle(X_n) \subseteq \bigcap_{\substack{x \in A \\ \max(x) \leq n}} A_x.$$

By stability, we can grab a linearly independent family $X$ such that $\mathrm{F}\triangle(X) \in p$, $\mathrm{F}\triangle(X) \subseteq \mathrm{F}\triangle(X_0)$ and for each $n < \omega$, there is a finite $F \subseteq X$ such that $\mathrm{F}\triangle(X \setminus F) \subseteq \mathrm{F}\triangle(X_n)$. We define a partition of $\omega$ into intervals $I_n = [a_n, a_{n+1})$ as follows: $a_0 = 0$ and, knowing $a_n$, we let

$$a_{n+1} = \max\{\min(x) \big| x \in X \text{ and } x \notin \mathrm{F}\triangle(X_{a_n})\} + 1.$$

Lemma 4.1 implies that, without loss of generality, min is injective in $X$. Hence, what we get is that, if $\min(x) \geq a_{n+1}$ for $x \in \mathrm{F}\triangle(X)$, then $x \in \mathrm{F}\triangle(X_{a_n})$. We now use Theorem 4.9 to get a condensation $Y \subseteq \mathrm{F}\triangle(X)$ such that $\mathrm{F}\triangle(Y) \in p$ and no interval from our partition is hit by both $\max[Y]$ and $\min[Y]$, and moreover there is always at least one "buffer" interval in between. We can again assume that min is injective on $Y$. The claim is that $\mathrm{F}\triangle(Y)^2_<$ is monochromatic in colour white. This is because, if $x, y \in \mathrm{F}\triangle(Y)$ are such that $\max(x) < \min(y)$ and

$a_n \leq \max(x) < a_{n+1}$, since there is always a "buffer" between $\max[Y]$ and $\min[Y]$ then we know that $\min(y) \geq a_{n+2}$, which by the previous observation implies that $y \in \mathrm{F}\triangle(X_{a_{n+1}}) \subseteq A_x$ since $\max(x) \leq a_{n+1}$. By the definition of $A_x$ this means that $(x, y)$ is coloured in white, and we are done. $\square$

It is quite interesting and surprising that, although the definition of stability resembles that of a P-point, it is equivalent (by Theorem 4.14) to something that is, in a sense, analogous to the defining property of a Ramsey ultrafilter. This will be even more apparent in the case of ordered union ultrafilters. We now observe that, even though the definition of stability does not explicitly state it, it is in fact possible to choose the common pseudocondensations to be disjoint families (respectively ordered families) if our ultrafilter is union (respectively ordered union).

**Theorem 4.15.** *If $p \in \mathbb{B}^*$ is a stable union ultrafilter (respectively stable ordered union ultrafilter), then whenever we have $\{A_n \big| n < \omega\} \subseteq p$, it is possible to choose a disjoint (respectively pairwise disjoint) common pseudocondensation $X$ as in the definition of stability. Moreover, if $p$ is stable ordered union and the sequence of $A_n$ is descending, then it is possible to choose the ordered pseudocondensation $X = \{x_n \big| n < \omega\}$ (where $\max(x_n) < \min(x_{n+1})$) in such a way that $(\forall n < \omega)(|X \cap (A_n \setminus A_{n+1})| \leq 1$ and, if we let $f(n)$ denote the unique $k$ such that $x_n \in A_k \setminus A_{k+1}$ then $f$ is increasing.*

*Proof.* Given $p$ and $\{A_n \big| n < \omega\}$ as in the hypotheses, we assume without loss of generality that the sequence of $A_n$ is decreasing and that $\bigcap_{n<\omega} A_n = \varnothing$. For $x \in A_0$ we define $f(x)$ to be the unique $k$ such that $x \in A_k \setminus A_{k+1}$ (we state the convention that $f(x) = -1$ for $x \notin A_0$). We now colour the pairs $(x, y) \in \mathbb{B}^2_<$ in colour white if $f(y) \leq f(x)$, and in colour black otherwise. By Theorem 4.14, $p$ has the Ramsey property for pairs, thus it is possible to pick a disjoint (respectively ordered) family $X$ such that $p \ni \mathrm{F}\triangle(X) \subseteq A_0$ and the set $\mathrm{F}\triangle(X)^2_<$ is monochromatic. If the colour for this monochromatic set was white then, given any $n < \omega$, the existence of an $x \in X \cap (A_n \setminus A_{n+1})$ would imply that for all $y \in \mathrm{F}\triangle(X)$ with $\min(y) > \max(x)$, we have $f(y) \leq f(x)$. This means that $y \notin A_{n+1}$, which is a contradiction because there are ultrafilter many such $y$ and $A_{n+1} \in p$. Thus the colour should be black and so for any $n < \omega$, if $x \in X \cap (A_n \setminus A_{n+1})$ then for $y \in X$, unless $y$ is one of the finitely many elements with $\min(y) \leq \max(x)$, it will be the case that $f(y) > n$ so $y \in A_{n+1}$ and we are done. For the "moreover" part, we just need to notice that, if $X$ is ordered then any two $x, y \in X$ are comparable. Since we argued that $\mathrm{F}\triangle(X)^2_<$ is monochromatic in colour black, this means that whenever $x, y \in X$ are distinct, it must be the case that $f(x) \neq f(y)$ and moreover, whether $f(x)$ or $f(y)$ is the largest is in agreement with whether $x$ or $y$ is the largest. $\square$

We are now ready to provide our announced characterization of stable ordered union ultrafilters in terms of games.

**Definition 4.16.** Given an ultrafilter $p \in \mathbb{B}^*$, we define a game $\mathscr{G}(p)$ as follows: in the $n$-th run, player I plays a set $A_n \in p$ and then player II responds with an element $x_n \in A_n$. After $\omega$ moves, we collect player II's moves into a family $X = \{x_n \big| n < \omega\}$, and player II wins if and only if $\mathrm{F}\triangle(X) \in p$.

We should first of all note that, if $p$ is nonprincipal, then it is impossible for player II to have a winning strategy, because we can imagine players I and II alternately playing two distinct games. Player I starts by playing any set $A_0 \in p$ for the first game, waits for player II's response $x_0 \in A_0$, and then plays the set $B_0 = A_0 \setminus \{x_0\}$ for the second game, and waits for player II's response $y_0 \in B_0$. Recursively, assume that the $n$-th move has been made by both players in both games, and the last sets played by player I were $A_n$ in the first game and $B_n$ in the second, while the collections of player's II moves are $\langle x_k \big| k \leq n\rangle$ for the first game and $\langle y_k \big| k \leq n\rangle$ for the second. We further assume as an induction hypothesis that $\mathrm{F}\triangle(\{y_k \big| k \leq n\})$ is disjoint from $\mathrm{F}\triangle(\{x_k \big| k \leq n\})$. Then we let player I play the set $A_{n+1} = A_n \setminus \mathrm{F}\triangle(\{x_k \big| k \leq n\} \cup \{y_k \big| k \leq n\}) \in p$ for the first game, wait for player II's response $x_{n+1} \in A_{n+1}$, and play the set $B_{n+1} = B_n \setminus \mathrm{F}\triangle(\{x_k \big| k \leq n+1\} \cup \{y_k \big| k \leq n) \in p$ in the second game (and wait for player II's response $y_{n+1} \in B_{n+1}$ afterwards). Note that in this way we get that $\mathrm{F}\triangle(\{y_k \big| k \leq n+1\})$ is disjoint from $\mathrm{F}\triangle(\{x_k \big| k \leq n+1\})$, so the induction hypothesis is preserved and we can continue. In other words, player I is forcing player II to play families $X$ for the first game, and $Y$ for the second, in such a way that $\mathrm{F}\triangle(X)$ is disjoint from $\mathrm{F}\triangle(Y)$. So regardless of any possible strategy that player II might be following, it is impossible for her to win both games, hence the strategy is not winning. Thus, whether the game $\mathscr{G}(p)$ is determined depends entirely on whether player I has a winning strategy. The following theorem, characterizing when such a strategy exists, is the main result of this section.

**Theorem 4.17.** *Let $p \in \mathbb{B}^*$ be an idempotent ultrafilter. Then, $p$ is a stable ordered union ultrafilter if and only if player I does not have a winning strategy in the game $\mathscr{G}(p)$.*

*Proof.* We first assume that $p$ is not a stable ordered union ultrafilter, and we will construct a winning strategy for player I. If $p$ fails to be ordered union, we can pick an $A \in p$ such that no ordered family $X$ with $\mathrm{F}\triangle(X) \subseteq A$ can satisfy $\mathrm{F}\triangle(X) \in p$. Now define a strategy for player I is as follows: in the first move she plays $A^\star$, and subsequently in the $n$-th move she plays

$$\left(\bigcap_{x \in \mathrm{F}\triangle(\langle x_k \big| k<n\rangle} x\blacktriangle A^\star\right) \cap \{x \in A^\star \big| \min(x) > \max\{\max(x_k)\big| k < n\}\} \in p,$$

where $\langle x_k \big| k < n\rangle$ is the sequence of previous moves of player II. This way we will ensure that, in the end, if $X = \{x_n \big| n < \omega\}$ is the collection of all moves of player II, then $X$ is an ordered family such that $\mathrm{F}\triangle(X) \subseteq A$, which implies that $\mathrm{F}\triangle(X) \notin p$ and so player II loses the game. Now, if $p$ is ordered union but fails to be stable, pick a sequence of $A_n \in p$ witnessing the failure of stability (this is, whenever $X$ is linearly independent and $\mathrm{F}\triangle(X) \in p$, there is an $n < \omega$ such that for no cofinite $Y \subseteq X$ do we have $\mathrm{F}\triangle(Y) \subseteq A_n$). Further, assume without loss of generality that each $A_n$ equals $\mathrm{F}\triangle(X_n)$ for some linearly independent family $X_n$. Our strategy dictates that player I plays $A_n = \mathrm{F}\triangle(X_n)$ in the $n$-th move. So regardless of what player II does, in the end she must have played a sequence $X = \{x_n \big| n < \omega\}$ satisfying, for every $n < \omega$, that $\{x_k \big| k \geq n\} \subseteq A_n = \mathrm{F}\triangle(X_n)$. Hence $\mathrm{F}\triangle(\{x_k \big| k \geq n\}) \subseteq \mathrm{F}\triangle(X_n) = A_n$, thus it must be the case that $\mathrm{F}\triangle(X) \notin p$ and so player I wins.

Conversely, we assume that $p$ is stable and we will show that no strategy for player I in the game $\mathscr{G}(p)$ can be winning. So let $s$ be a strategy for player I, this is, $s$ is a function that takes finite sequences $\vec{x} = \langle x_k \big| k < n \rangle$ (the sequence of moves that player II has made so far) as input, and returns some element of $p$ as output (the element that player I should play at that move, according to the strategy). Notice that, if $s$ is winning and we modify $s$ into an $s'$ such that, for every $\vec{x}$, we have $s'(\vec{x}) \subseteq s(\vec{x})$, then $s'$ is still a winning strategy (since all we are doing is restricting the possibilities for player II, who already has no hope of winning). Thus, we modify the strategy $s$ as follows: First of all, given an $n < \omega$ we let $S_n$ be the set of all finite sequences $\vec{x} = \langle x_i \big| i < k \rangle \in \mathrm{dom}(s)$ such that $\max\left(\bigcup_{i<k} x_i\right) = n$,

and we let $A_n = \bigcap_{\vec{x} \in \bigcup_{k \le n} S_n} s(\vec{x}) \in p$ (notice that each $S_n$ is finite). We further shrink $A_n$ to something of the form $\mathrm{F}\triangle(X_n) \in p$, and if we do this recursively we can ensure that $\mathrm{F}\triangle(X_{n+1}) \subseteq \mathrm{F}\triangle(X_n)$. We now define the "shrunk" strategy $s'$ by $s'(\vec{x}) = \mathrm{F}\triangle(X_n)$ whenever $\vec{x} \in S_n$. By the above observation, if we prove that $s'$ is not winning, we will be able to conclude that $s$ is not winning either and we will be done.

Hence we will prove that the strategy $s'$ is not winning. We first pick, by Theorem 4.15, an ordered family $X = \{x_n \big| n < \omega\}$ (where $\max(x_n) < \min(x_{n+1})$) such that $\mathrm{F}\triangle(X) \in p$, $\mathrm{F}\triangle(X) \subseteq \mathrm{F}\triangle(X_0)$ and, if we define $f(x)$ to be the unique $k$ with $x \in \mathrm{F}\triangle(X_k) \setminus \mathrm{F}\triangle(X_{k+1})$ for all $x \in \mathrm{F}\triangle(X)$, then $n < m$ implies $f(x_n) < f(x_m)$. At this point, it is worth noting that for $x = \bigtriangleup_{i \in a} x_i \in \mathrm{F}\triangle(X)$, we will have that $f(x) = f(x_{\min(a)})$. We now define a partition of $\omega$ into intervals $I_n = [a_n, a_{n+1})$ as follows: $a_0 = 0$, $a_1 = \max(x_0) + 1$ and, if we know $a_n$, we let

$$a_{n+1} = \max\left(\bigcup_{\substack{i<\omega \\ f(x_i) \le a_n}} x_i\right) + 1.$$

Notice that, by the previous observation, if $x = \bigtriangleup_{i \in a} x_i \in \mathrm{F}\triangle(X)$ and $\min(x) \ge a_{n+1}$ then $\min(a)$ is big enough so that $f(x) = f(x_{\min(a)}) > a_n$ and so, in particular, $x \in \mathrm{F}\triangle(X_{a_n})$. We now use Theorem 4.9 to get an ordered condensation $Y = \{y_n \big| n < \omega\} \subseteq \mathrm{F}\triangle(X)$ (here $\max(y_n) < \min(y_{n+1})$) such that $\mathrm{F}\triangle(Y) \in p$ and no interval from our partition is hit by both $\max[Y]$ and $\min[Y]$, and moreover there is always at least one "buffer" interval in between. We claim that player II can play the family $Y$ in response to strategy $s'$, which would then show that $s'$ is not winning. Certainly player II can respond $y_0 \in \mathrm{F}\triangle(X) \subseteq \mathrm{F}\triangle(X_0)$ in the first move. Assuming that player II has been successfully able to play the sequence $\langle y_k \big| k < n\rangle$, we let $m$ be such that $\langle y_k \big| k < n\rangle \in S_m$, so that player I responds by playing $\mathrm{F}\triangle(X_m)$. This means that $\max(y_{n-1}) = m$, since $\max[Y]$ and $\min[Y]$ cannot simultaneously hit the same interval from our partition and moreover there is always at least one "buffer" interval in between, we conclude that if $a_l \leq m < a_{l+1}$ then $\min(y_n) \geq a_{l+2}$. By the previous observation, this implies that $y_n \in \mathrm{F}\triangle(X_{a_{l+1}}) \subseteq \mathrm{F}\triangle(X_m)$ and so player II can successfully play the element $y_n$. This finishes the proof. □

After this characterization, we finish the section by showing the construction of stable ordered union ultrafilters, which we will need to use in the future. Interestingly, in order to ensure stability of a strongly summable ultrafilter while constructing it, it seems that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is not strong enough of an assumption (unlike the case where we drop the stability requirement).

**Theorem 4.18.** *Assuming $\mathfrak{p} = \mathfrak{c}$, there exists a stable ordered union ultrafilter. Moreover, every family $\{\mathrm{F}\triangle(X_\alpha)\big|\alpha < \kappa\}$ with the strong finite intersection property such that $\kappa < \mathfrak{p}$, where each $X_\alpha$ is an ordered family, can be extended to a stable ordered union ultrafilter.*

*Proof.* Grab such a family and enumerate $\langle A_\alpha \big| \kappa \leq \alpha < \mathfrak{c} \rangle$ all subsets of $\mathbb{B}$. We will recursively choose some more ordered families $X_\alpha$, for $\kappa \leq \alpha < \mathfrak{c}$, that satisfy $(\forall \beta < \alpha)(X_\alpha \subseteq^* \mathrm{F}\triangle(X_\beta))$ and such that for some $B \in \{A_\alpha, \mathbb{B} \setminus A_\alpha\}$, $\mathrm{F}\triangle(X) \subseteq B$. If we furthermore ensure that $\{\mathrm{F}\triangle(X_\beta) \big| \beta \leq \alpha\}$ has the strong finite intersection property, then in the end clearly the filter $p$ generated by $\{\mathrm{F}\triangle(X_\alpha) \big| \alpha < \mathfrak{c}\}$ will be as desired.

So assume that we already know $X_\beta$ for $\beta < \alpha$. Let $q$ be an idempotent ultrafilter extending $\{\mathrm{F}\triangle(X_\beta) \big| \beta < \alpha\}$ (which exists since the collection of ultrafilters extending this family is a closed subsemigroup of $\mathbb{B}^*$). Pick $B \in \{A_\alpha, \mathbb{B} \setminus A_\alpha\}$ such that $B \in q$. We will let $\mathbb{P}$ be the forcing notion whose conditions are all those pairs $(a, A)$ where $a$ is a finite linearly independent subset of $\mathbb{B}$ such that $\mathrm{F}\triangle(a) \subseteq B^\star$, and $A \in q$. The order would be $(a, A) \leq (a', A')$ iff $A \cup (a \setminus a') \subseteq A'$. Notice that for each $\beta < \alpha$, the set

$$D_\beta = \{(a, A) \in \mathbb{P} \big| A \subseteq \mathrm{F}\triangle(X_\beta)\}$$

is dense in $\mathbb{P}$. Since we have $|\alpha| \leq \alpha < \mathfrak{c} = \mathfrak{p}$ many such dense sets, it is possible to find a filter meeting them all. Quite straightforwardly this filter gives rise to an $X_\alpha$ satisfying all of the requirements. $\square$

## 4.3 Strongly Summable Ultrafilters and Small $\mathrm{cov}(\mathcal{M})$

In this section we will show that the existence of strongly summable ultrafilters on any abelian group is consistent with $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$. It follows immediately from Corollary 3.28 that every strongly summable ultrafilter on an abelian group is additively isomorphic to a strongly summable ultrafilter on the Boolean group $\mathbb{B}$. In fact, if there is a union ultrafilter then there are strongly summable ultrafilters on every abelian group $G$. For this reason, we will focus in this section on constructing union ultrafilters in the models that we consider. We use two different kinds of forcing notions The first one will be a variant of the Prikry-Mathias forcing, with side conditions on a strongly summable ultrafilter.

**Definition 4.19.** Given a strongly summable ultrafilter $p$ on $\mathbb{B}$, we denote by $\mathbb{M}(p)$ (our own version of the Prikry-Mathias forcing with side conditions in $p$) the partial order whose elements are of all pairs $(a, A)$ such that $a \in [\mathbb{B}]^{<\omega}$ is linearly independent and $A \in p$; and we say that $(a, A) \leq (b, B)$ iff $b \subseteq a$ and $A \cup (a \setminus b) \subseteq B$. We call the first coordinate $a$ of a condition $(a, A) \in \mathbb{M}(p)$ the **stem** of the condition.

The only difference with the usual Prikry-Mathias forcing is that we demand that the stem is a linearly independent set (this is not essential, but it simplifies the exposition). It is clear that any two conditions with the same stem are compatible, hence the preorder $\mathbb{M}(p)$ is $\sigma$-centred (and hence c.c.c.).

Now we work out some simplifications. First note that, since $p$ is strongly summable, conditions have the form $(a, \mathrm{F}\triangle(X))$ on a dense set (one can also demand that $X$ is linearly independent, and even that for every $x \in a$ and every $y \in X$, one has $\max(x) < \min(y)$), thus we may as well look at conditions of that form only. Now, when comparing two such conditions $(a, \mathrm{F}\triangle(X))$ and $(b, \mathrm{F}\triangle(Y))$, we observe that the former extends the latter iff $b \subseteq a$ and $X \cup (a \setminus b) \subseteq \mathrm{F}\triangle(Y)$. Hence, it is also possible to think of a condition in $\mathbb{M}(p)$ as given by the information $(a, X)$, where $a$ and $X$ are two linearly independet subsets of $\mathbb{B}$, the first one finite and the second infinite, such that $\mathrm{F}\triangle(X) \in p$; and that $(a, X) \leq (b, Y)$ iff $b \subseteq a$ and $X \cup (a \setminus b) \subseteq \mathrm{F}\triangle(Y)$.

We let $\mathring{X}$ be a $\mathbb{M}(p)$-name for the union of all stems of elements of the generic filter, which we call the *generic linearly independent subset of* $\mathbb{B}$ *added by* $\mathbb{M}(p)$. Notice that any condition $(b, Y)$ forces that "$\mathring{X} \setminus \check{b} \subseteq \mathrm{F}\triangle(\check{Y})$". Hence for every $A \in p$ from the ground model, in the generic extension $V[G]$ we have that $\mathrm{F}\triangle(X \setminus a) \subseteq A$ for some finite $a \subseteq X$. (since it is dense to have conditions $(b, Y)$ with $\mathrm{F}\triangle(Y) \subseteq A$). Thus, in $V[G]$, the countable family $\{\mathrm{F}\triangle(X \setminus a) \big| a \in [X]^{<\omega}\}$ generates a filter $\mathscr{F}$ with the property that every ground model set $A \subseteq \mathbb{B}$ is either in $\mathscr{F}$ or in the dual ideal $\mathscr{F}^*$.

In what follows we urge the reader to keep in mind that, as a particular case of Theorem 2.8 from Chapter 2, under the assumption that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ we have that, if $\mathscr{F}$ is a filter generated by a family of the form $\{\mathrm{F}\triangle(X_\alpha)\big|\alpha < \lambda\}$ for some $\lambda < \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ and all $X_\alpha \subseteq \mathbb{B}$ linearly independent sets then there is a strongly summable ultrafilter $p$ extending $\mathscr{F}$ (the particular case of this statement that refers to union ultrafilters was first proved in [10]). We are now in a good position to tackle the iteration of forcings of the form $\mathbb{M}(p)$.

**Theorem 4.20.** *Let $\lambda, \kappa$ be two regular cardinals such that $\omega_1 \leq \lambda < \kappa = \kappa^\omega$ (in the ground model). Then there is a finite support iteration of forcing notions of the form $\mathbb{M}(p)$ such that the generic extension satisfies that $\mathrm{cov}(\mathcal{M}) = \lambda < \kappa = \mathfrak{c}$ and there exist strongly summable ultrafilters on $\mathbb{B}$.*

*Proof.* Define the FS iteration iteration $\mathbb{P} = \mathbb{P}_\lambda$ with iterands $\mathring{\mathbb{Q}}_\alpha$ (this is, for each $\alpha < \lambda$ we let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathring{\mathbb{Q}}_\alpha$ and if $\alpha = \bigcup \alpha$ then $\mathbb{P}_\alpha$ is the direct limit of the $\mathbb{P}_\xi$ for $\xi < \alpha$) as follows. $\mathbb{P}_0 = \mathbb{Q}_0 = \mathrm{Fn}(\kappa, 2) \star \mathbb{M}(\mathring{p}_0)$, where $\mathring{p}_0$ is a $\mathrm{Fn}(\kappa, 2)$-name such that

$$\Vdash \text{“}\mathring{p}_0 \text{ is a strongly summable ultrafilter on } \check{\mathbb{B}}\text{”}$$

(notice that after forcing with $\mathrm{Fn}(\kappa, 2)$ we have that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \kappa$, hence such a $\mathring{p}_0$ is guaranteed to exist). Then we let $\mathring{X}_0$ be the $\mathbb{P}_0$-name for the generic linearly independent set added by $\mathbb{M}(\mathring{p}_0)$.

Now for $\alpha < \lambda$, recursively define names $\mathring{p}_\alpha, \mathring{\mathbb{Q}}_\alpha, \mathring{X}_\alpha$ (in that order) satisfying:

$$\mathbb{P}_\alpha \star \mathrm{Fn}(\kappa, 2) \Vdash \text{“}\mathring{p}_\alpha \text{ is a strongly summable ultrafilter}$$

$$\text{extending } \{\mathrm{F}\triangle(\mathring{X}_\xi \setminus a) \big| a \in [\mathring{X}_\xi]^{<\omega} \wedge \xi < \alpha\}\text{”},$$

(which is possible because $\mathbb{P}_\alpha \star \mathrm{Fn}(\kappa, 2) \Vdash$ “$\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \check{\kappa}$”),

$$\mathbb{P}_\alpha \Vdash \text{“}\mathring{\mathbb{Q}}_\alpha = \mathrm{Fn}(\kappa, 2) \star \mathbb{M}(\mathring{p}_\alpha)\text{”},$$

and $\mathring{X}_\alpha$ is the name for the generic linearly independent set added to $V^{\mathbb{P}_\alpha \star \mathrm{Fn}(\kappa,2)}$ by $\mathbb{M}(\mathring{p}_\alpha)$. This defines our iteration. (Informally we might phrase this argument as follows: at each step, we first add $\kappa$-many Cohen reals, to ensure that $\mathfrak{c} = \mathrm{cov}(\mathcal{M}) = \kappa$, and after that we use Lemma 2.8 to extend “everything we've got so far” to a further strongly summable ultrafilter, which we then plug into our version of Mathias-Prikry and force with that. Lather, rinse, repeat... $\lambda$ many times.)

In the end (i.e. at stage $\lambda$), since every real added by $\mathbb{P}_\lambda$ actually appears at an intermediate stage $\alpha$ (after which the next generic linearly independent set $X_\alpha$ diagonalizes it), we get that in $V^{\mathbb{P}_\lambda}$, the family

$$\{\mathrm{F}\triangle(X_\alpha \setminus a) \big| \alpha < \lambda \wedge a \in [X_\alpha]^{<\omega}\}$$

generates an ultrafilter $p$, which is by definition strongly summable (generated by $\mathrm{F}\triangle$-sets).

The proof finishes by noticing that, in $V^{\mathbb{P}_\lambda}$, we have that $\mathrm{cov}(\mathcal{M}) = \lambda < \kappa = \mathfrak{c}$. It is certainly easy to see that $\mathfrak{c} = \kappa$, and in order to calculate the value of $\mathrm{cov}(\mathcal{M})$, we first notice that the ultrafilter $p$, being generated by $\lambda$ many sets, is a witness that $\mathfrak{u} \leq \lambda$ (recall that the cardinal invariant $\mathfrak{u}$ is defined as the least cardinality of a family generating an ultrafilter, and it is well-known that the inequality $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{u}$ is provable in $\mathsf{ZFC}$). On the other hand, the fact that our iteration of length $\lambda$ adds Cohen reals cofinally often (both because of the finite support, and because each iterand has "add $\kappa$-many Cohen reals" as a factor) implies that $\lambda \leq \mathrm{cov}(\mathcal{M})$ and hence we conclude that $\mathrm{cov}(\mathcal{M}) = \mathfrak{u} = \lambda$ in our generic extension. □

We will now start working with a different forcing notion, which will also use an ultrafilter as a parameter, and which behaves nicely not only when said ultrafilter is strongly summable, but even when it is just an idempotent ultrafilter. This represents a slight advantage with respect to the previous construction because the existence of idempotent ultrafilters is a $\mathsf{ZFC}$ theorem and so we will not need to add Cohen reals at each stage.

**Definition 4.21.** Given an ultrafilter $p$ on $\mathbb{B}$, we define the **ultraLaver forcing on** $p$ to be the partially ordered set $\mathbb{L}(p)$ whose elements are subtrees $T$ of $\mathbb{B}^{<\omega}$ (that is, $T$ is closed under initial segments) that have a stem $s(T)$ (this is, every node $t \in T$ is comparable with $s(T)$) such that "the branching is in $p$ above $s(T)$" (which means: for every $t \in T$ such that $t \geq s(T)$, the set of immediate successors $\mathrm{succ}_T(t) = \{x \in \mathbb{B} \big| t \frown x \in T\} \in p$). The ordering is given by $T' \leq T$ iff $T' \subseteq T$.

Normally this forcing notion is defined on $\omega$ rather than $\mathbb{B}$, and with a different ultrafilter for each node, but the definition as we stated it above is what we will need for our purpose. The following are well-known properties of ultraLaver forcing (see for example [14, Section 1A]), and are also not terribly difficult to prove.

- $\mathbb{L}(P)$ is $\sigma$-centred (hence c.c.c. and proper).
- $\mathbb{L}(P)$ has the **pure decision property**: given any statement $\varphi$ in the forcing language and any condition $T \in \mathbb{L}(P)$, it is possible to find a *pure extension* $T' \leq^* T$ (this is, $T' \leq T$ and $s(T') = s(T)$) deciding $\varphi$ (i.e. either $T' \Vdash \varphi$ or $T' \nVdash \varphi$).
- As a direct consequence of the previous point, whenever $F$ is a finite set in $V$ and $\mathring{x}$ is an $\mathbb{L}(p)$-name such that some condition $T$ forces $T \Vdash$ "$\mathring{x} \in \check{F}$", there is a pure extension $T' \leq^* T$ and an element $y \in F$ such that $T' \Vdash$ "$\mathring{x} = \check{y}$".

Note that, at this point, we still do not assume any special property of $p$ other than its being an ultrafilter. The following lemma shows that the situation becomes quite interesting when $p$ is idempotent. We will denote by $\mathring{X}$ the $\mathbb{L}(p)$-name for the generic subset that arises from the generic filter (which is the union of all the stems of, or equivalently the intersection of all conditions in, the generic filter). Also, recall that, if $p$ is idempotent and $A \in p$, then we have that $A^\star = \{x \in A \big| x \blacktriangle A \in p\} \in p$ and, moreover, for each $x \in A^\star$ it is the case that $x \blacktriangle A^\star \in p$ (i.e. $(A^\star)^\star = A^\star$).

**Lemma 4.22.** *Let $p \in \mathbb{B}^*$ be an idempotent ultrafilter, and $A \in p$ a ground model set. Then, in the generic extension obtained by forcing with $\mathbb{L}(p)$, there is a finite set $a$ such that $\mathrm{F}\triangle(X \setminus a) \subseteq A$.*

*Proof.* It suffices to prove that every condition $T$ can be extended to a condition $T'$ which will force "$\mathrm{F}\triangle(\mathring{X} \setminus s(T')) \subseteq \check{A}$". In order to do that, we reprove Hindman's theorem along the condition $T$, which means we recursively define the levels $(T')_n$ of the extension $T'$. First we let $s(T') = s(T)$. Assume that we have defined the $n$-th level above the stem $(T')_{|s(T)|+n}$. Then the $n+1$-st. level above the stem is given by specifying that, for every $t = s(T) \frown \langle x_0, \dots, x_n \rangle \in (T')_{|s(T)|+n}$, with the additional inductive hypothesis that $\mathrm{F}\triangle(\{x_0, \dots, x_n\}) \subseteq A^\star$, we let

$$\mathrm{succ}_{T'}(t) = \mathrm{succ}_T(t) \cap A^\star \cap \left( \bigcap_{x \in \mathrm{F}\triangle(\{x_0,\dots,x_n\})} x \blacktriangle A^\star \right)$$

and check that the inductive hypotheses still hold for all of the new nodes $t \frown x$, so that the construction can continue. What we are doing is basically repeating Galvin-Glazer's argument for Hindman's theorem above the stem of $T$, and just as in that argument, it is easy to see that for every branch $f$ of $T'$ we have that $\mathrm{F}\triangle(\{f \upharpoonright |s(T)| + n \big| n < \omega\}) \subseteq A$. Note that this also implies that

$$T' \Vdash \text{"}\mathrm{F}\triangle(\{X \setminus s(T)\}) \subseteq \check{A}\text{"}$$

(since for every finite subset $a \subseteq \omega$, there is an extension $T'' \leq T'$ deciding that the generic set $\mathring{X}$ coincides with some ground-model branch $f$ of $T'$ up to the

$|s(T)| + \max(a)$-th element), and we are done. $\square$

Once we have the previous lemma under our belt, we are ready to state and prove a result which is analogous to Theorem 4.20 except that it uses ultraLaver forcings with idempotent ultrafilters as parameter, instead of our variation of Prikry-Mathias forcing with a strongly summable ultrafilter.

**Theorem 4.23.** *Let $\lambda, \kappa$ be two regular cardinals such that $\omega_1 \leq \lambda < \kappa = \kappa^\omega$ (in the ground model). Then there is a finite support iteration of forcing notions of the form $\mathbb{L}(p)$ such that the generic extension satisfies that $\mathrm{cov}(\mathcal{M}) = \lambda < \kappa = \mathfrak{c}$ and there exist strongly summable ultrafilters on $\mathbb{B}$.*

*Proof.* The FS iteration iteration $\mathbb{P} = \mathbb{P}_\lambda$ with iterands $\mathring{\mathbb{Q}}_\alpha$ is given by recursively defining the names $\mathring{p}_\alpha, \mathring{\mathbb{Q}}_\alpha, \mathring{X}_\alpha$ (in that order) such that:

$\mathbb{P}_\alpha \Vdash$ "$\mathring{p}_\alpha$ is an idempotent ultrafilter extending $\{\mathrm{F}\triangle(\mathring{X}_\xi \setminus a)\big| a \in [\mathring{X}_\xi]^{<\omega} \wedge \xi < \alpha\}$",

(which is always possible because of Ellis's Lemma),

$$\mathbb{P}_\alpha \Vdash \text{“}\mathring{\mathbb{Q}}_\alpha = \mathbb{L}(\mathring{p}_\alpha)\text{”},$$

and $\mathring{X}_\alpha$ is the name for the generic linearly independent set added to $V^{\mathbb{P}_\alpha}$ by $\mathbb{L}(\mathring{p}_\alpha)$.

At stage $\lambda$, every real added by $\mathbb{P}_\lambda$ actually appears at an intermediate stage $\alpha$ (after which the next generic linearly independent set $X_\alpha$ diagonalizes it), hence in $V^{\mathbb{P}_\lambda}$ the family

$$\{\mathrm{F}\triangle(X_\alpha \setminus a)\big|\alpha < \lambda \wedge a \in [X_\alpha]^{<\omega}\}$$

generates an ultrafilter $p$, which is by definition strongly summable.

Finally, note that in $V^{\mathbb{P}_\lambda}$, we have that $\mathrm{cov}(\mathcal{M}) = \lambda < \kappa = \mathfrak{c}$ (the argument for this is exactly as in Theorem 4.20). $\square$

Both ultraLaver forcing and our version of the Prikry-Mathias forcing could conceivably be iterated with countable support, since they are $\sigma$-centred and hence proper. However, in order for iterations of proper forcings with countable support to yield something interesting (i.e. models of $\neg\mathsf{CH}$), it is necessary to do iterations of length $\omega_2$, which leaves us with a value of $\mathfrak{c} = \omega_2$ at the end. Thus we will not be able to get a model with small $\mathrm{cov}(\mathcal{M})$ unless we make sure that the forcing that we are iterating, as well as its iterations, do not add any Cohen reals. Unfortunately this means that our version of Prikry-Mathias forcing is bound to yield failure, as it is easy to see that the forcing notion $\mathfrak{M}(p)$ adds Cohen reals if and only if $p$ is not a P-point, and strongly summable ultrafilters (or even idempotent ultrafilters, for that matter) are never P-points. We will, however, be able to profitably iterate ultraLaver forcing with countable support, as we now proceed to explain.

**Definition 4.24.** A forcing notion $\mathbb{P}$ satisfies the **Laver property** if whenever $g : \omega \longrightarrow \omega$ (in the ground model), $q$ is a condition, and $\mathring{f}$ is a $\mathbb{P}$-name such that

$$q \Vdash \text{“}\mathring{f} : \omega \longrightarrow \omega \text{ and } \mathring{f} \leq \check{g}\text{”},$$

there is $F : \omega \longrightarrow [\omega]^{<\omega}$ and $r \leq q$ such that for every $n < \omega$, $|F(n)| \leq 2^n$ and

$r \Vdash$ "$\mathring{f}(\check{n}) \in \check{F}(\check{n})$".

The Laver property is important because of two reasons. The first is that it is preserved under CS iterations, and the second is that, whenever $\mathbb{P}$ has the Laver property, it does not add any Cohen reals [1, 34]. Hence if we force with a CS iteration of forcings satisfying the Laver property, $\mathrm{cov}(\mathcal{M})$ in the generic extension will have the same value that it used to have in the ground model (if our ground model satisfies $\mathsf{CH}$; then after forcing with a forcing notion that satisfies the Laver property, we will get that $\mathrm{cov}(\mathcal{M}) = \omega_1$). The following theorem establishes a condition on the strongly summable ultrafilter $p$ that will ensure that $\mathbb{L}(p)$ has the Laver property. The author has to admit that the proof of the following theorem is his favourite from among all of the proofs that appear in this dissertation.

**Theorem 4.25.** *If $p$ is a stable ordered union ultrafilter, then $\mathbb{L}(p)$ satisfies the Laver property.*

*Proof.* Let $T \in \mathbb{L}(p)$, $g : \omega \longrightarrow \omega$ and $\mathring{f} \in V^{\mathbb{L}(p)}$ be such that

$$T \Vdash \text{“}\mathring{f} : \check{\omega} \longrightarrow \check{\omega} \text{ and } \mathring{f} \leq \check{g}\text{”}$$

We will recursively construct an extension $T' \leq T$ that will satisfy that

$$T' \Vdash \text{“}(\forall n < \check{\omega})(\mathring{f}(n) \in \check{F}(n))\text{”}$$

for some ground-model function $F : \omega \longrightarrow [\omega]^{<\omega}$ such that for each $n < \omega$, $|F(n)| \leq 2^n$. We first let $h(n)$ be the number of finite sequences of natural numbers

$\langle m_1, \ldots, m_k \rangle$ that satisfy $\sum_{i=1}^{k} \lfloor \log_2(m_i) \rfloor = n$, and we pick and fix any increasing sequence $\langle k_n \big| n < \omega \rangle$ satisfying that $2^{k_n} \geq 2^{n+1} h(n+1)$. We moreover use the fact that for every Laver condition, there is a natural order-preserving bijection between $\omega^{<\omega}$ and the nodes of the condition above the stem.

We now define $T'$ by induction on the nodes. This is, if we have already decided that a certain $t \in T$ will belong to $T'$, we will show how to pick the set of immediate succesors $\mathrm{succ}_{T'}(t)$. For this, we will assume that not only have we decided that $t \in T'$, but we have also decided which will be the sequence $\langle m_1, \ldots, m_k \rangle$ associated to $t$ under the aforementioned order-preserving bijection (between $\omega^{<\omega}$ and the nodes of $T'$ above the stem) and we also assume that we have picked an auxiliary condition $T_t \leq^* T \upharpoonright t$ (here $T \upharpoonright t$ denotes the condition $\{s \in T \big| s \text{ is comparable with } t\}$) which decides the value of $f \upharpoonright k_n$, where $n = \sum_{i=1}^{k} \lfloor \log_2(m_i) \rfloor$.

Now we play the game $\mathscr{G}(p)$. The first thing to do is shrink, if necessary, the set $\mathrm{succ}_T(t)$ to something of the form $\mathrm{F}\triangle(Y)$, so that it is closed under $\triangle$. This way, at the end of the game we will be able to collect player II's moves $X = \{x_n \big| n < \omega\}$ and we will define $\mathrm{succ}_{T'}(t) = \mathrm{F}\triangle(X)$. Player I will adhere to the following strategy. First extend, for each $s \in \mathrm{succ}_T(t) = \mathrm{F}\triangle(Y)$, the condition $T_t \upharpoonright s$ to some condition $T_s^0$ with the same stem deciding the value of $\mathring{f} \upharpoonright k_{n+1}$ to be a certain $f_s^0$. The hypothesis that $T \Vdash$ "$\mathring{f} \leq \check{g}$" implies that there are only finitely many possible $f_s^0$, so there is a set $A_0 \in p$ such that all $T_s^0$ for $s \in A_1$ decide $\mathring{f} \upharpoonright k_{n+1}$ to be the same $f_0$. Player I starts by playing this set, and waits for player II to play some $x_0 \in A_0$. The auxiliary condition associated to $x_0$ in order to continue with the induction later on, will be $T^0_{x_0}$. We now extend, for each $s \in \mathrm{F}\triangle(Y) \setminus \{x_0\}$, the condition $T^0_s$ to some further condition $T^1_s$ with the same stem which decides the value of $\mathring{f} \upharpoonright k_{n+2}$ to be some $f^1_s$. Now (and here is the interesting twist) to each such $s$ we associate the couple $\langle f^1_s, f^1_{s\triangle x_0}\rangle$, and since there are only finitely many possibilities for such a couple, there exists a set $A_1 \in p$ such that for all $s \in A_1$ the aforementioned couple is constantly some fixed couple $\langle f_1, f_2\rangle$. Then we let player I play the set $A_1$ and wait for player II's response $x_1 \in A_1$. We will let the auxiliary conditions associated to $x_1$ and $x_0 \triangle x_1$ be $T^1_{x_1}$ and $T^1_{x_0\triangle x_1}$, respectively.

In general, if we are about to play the $m$-th inning of the game $\mathscr{G}(p)$, we assume that we know $\vec{x} = \langle x_i \big| i < m\rangle$ and the auxiliary conditions associated to each $x \in \mathrm{FS}(\vec{x})$, which decide the value of $\mathring{f} \upharpoonright k_{n+\max\{i<m|x_i\in\mathrm{supp}_{\vec{x}}(x)\}+1}$. We now extend, for each $s \in \mathrm{F}\triangle(Y) \setminus \mathrm{F}\triangle(\vec{s})$, the condition $A^{m-1}_s$ to some pure extension $A^m_s$ which decides the value of $\mathring{f} \upharpoonright k_{n+m+1}$ to be a certain $f^m_s$. Since there are only finitely many possibilities for the vector

$$\langle f^m_s\rangle \frown \langle f^m_{s\triangle x} \big| x \in \mathrm{F}\triangle(\vec{x})\rangle,$$

then there exists an $A_m \in p$ such that for all $s \in A_m$, the aforementioned vector is some fixed $\langle f_{2^{m-1}+1}, \dots, f_{2^m}\rangle$. We let player I play the set $A_m$ and wait for player II's response $x_m \in A_m$, and we establish that the auxiliary condition associated to

$x_m$ is $T^m_{x_m}$ and the one associated to $x_m \triangle x$ will be $T^m_{x_m \triangle x}$, for each $x \in \mathrm{F}\triangle(\vec{x})$.

In the end, since the described strategy cannot be winning, there is a possibility for player II to have won the game, i.e. $\mathrm{F}\triangle(X) \in p$. For each $x \in \mathrm{F}\triangle(X)$, we let the sequence associated to $t \frown \langle x \rangle$ (for the order-preserving bijection with $[\omega]^{<\omega}$) be $\langle m_1, \dots, m_k \rangle \frown \langle \sum_{x_i \in \mathrm{supp}_X(x)} 2^i \rangle$, and the induction can continue. It is important to note that, for every $x \in \mathrm{F}\triangle(X)$ and every $j \leq \max\{i < \omega | x_i \in \mathrm{supp}_X(x)\}$, the auxiliary condition $T_{t \frown \langle x \rangle}$ forces the value of $f \upharpoonright k_{n+j+1}$ to agree with some entry of the vector $\langle f_{2^{t-1}+1}, \dots, f_{2^t} \rangle$ (where $t = \lfloor \log_2(j) \rfloor$) which was chosen during the $t$-th run of the game $\mathscr{G}(p)$.

This way we get our condition $T' \leq T$ (in fact, $T'$ and $T$ have the same stem). It is straightforward to check that, given any $n < \omega$, if $k_{i-1} \leq n < k_i$ (with the convention that $k_{-1} = 0$) then $T' \Vdash$ "$\mathring{f}(\check{n}) \in \check{F(n)}$", where $F(n)$ is the collection of all entries from the vectors $\langle f_{2^i+1}, \dots, f_{2^{i+1}} \rangle$ obtained when doing the induction over a node $t \in T'$ whose associated sequence (under the bijection with $[\omega]^{<\omega}$) is some $\langle m_1, \dots, m_k \rangle$ satisfying $\sum_{j=1}^k \lfloor \log_2(m_k) \rfloor = i$. Since there are only $h(i)$ many such sequences, it follows that $|F(n)| \leq 2^i h(i) \leq 2^{k_{i-1}} \leq 2^n$. □

**Theorem 4.26.** *If we have* CH *(in the ground model), then there exists a countable support iteration of forcings of the form* $\mathbb{L}(p)$ *(where each of these* $p$ *is a stable ordered union ultrafilter) such that, in the generic extension, we have that* $\mathrm{cov}(\mathcal{M}) = \omega_1 < \omega_2 = \mathfrak{c}$ *and there exists a union ultrafilter.*

*Proof.* We define the CS iteration $\langle \mathbb{P}_\alpha \big| \alpha < \omega_2 \rangle$ with iterands $\mathring{\mathbb{Q}}_\alpha$ (i.e. for every $\alpha < \omega_2$ we have that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathring{\mathbb{Q}}_\alpha$ and, if $\alpha = \bigcup \alpha$, then $\mathbb{P}_\alpha$ is the direct or inverse limit, respectively, of the $\mathbb{P}_\xi$ ($\xi < \alpha$), according to whether $\alpha$ has uncountable or countable cofinality) in such a way that, for each $\alpha < \omega_2$, $\mathring{\mathbb{Q}}_\alpha$ is forced to be $\mathbb{L}(\mathring{p}_\alpha)$ for some specific (name of a) stable ordered union ultrafilter $\mathring{p}_\alpha$ from $V^{\mathbb{P}_\alpha}$. The main issue is how to choose the (names for) ultrafilters $\mathring{p}_\alpha$. We do this recursively. For every $\alpha < \omega_2$, we denote by $\mathring{X}_\alpha$ the $\mathbb{P}_{\alpha+1}$-name for the generic linearly independent set added by the last factor $\mathbb{L}(\mathring{p}_\alpha)$. Then we let $\mathring{p}_{\alpha+1}$ be any (name for a) stable ordered union ultrafilter extending $\{\mathrm{F} \triangle (X_\alpha \setminus F) \big| F \in [X_\alpha]^{<\omega}\}$ (which exists by Theorem 4.18). The only problem remaining is how to define $\mathring{p}_\alpha$ for limit $\alpha$. If $\alpha$ has uncountable cofinality, then every real in $V^{\mathbb{P}_\alpha}$ has already appeared at some intermediate stage and hence the family $\{\mathrm{F} \triangle (X_\xi \setminus F) \big| \xi < \alpha \wedge F \in [X_\xi]^{<\omega}\}$ generates an ultrafilter, which is the one that we take to be $p_\alpha$. Now, for $\alpha$ of cofinality $\omega$, we pick a cofinal sequence $\langle \alpha_n \big| n < \omega \rangle$ converging to $\alpha$ and let $p_\alpha$ be any stable ordered union ultrafilter that extends the filter $\{\mathrm{F} \triangle (X_{\alpha_n} \setminus F) \big| n < \omega \wedge F \in [X_{\alpha_n}]^{<\omega}\}$ (again by Theorem 4.18). Note that the construction is performed in such a way that every $p_\alpha$ extends the filter $p_\xi$ whenever $\xi < \alpha$. At the end, $p_{\omega_2}$ is the witness to the existence of a union ultrafilter in the final extension. And $\mathrm{cov}(\mathcal{M}) = \omega_1$ because by the previous lemma, $\mathbb{P}_{\omega_2}$ does not add Cohen reals. □

The reader might wonder how do the three different models presented here differ from each other. The first thing to notice is that FS iterations are much more flexible in the values that they allow for both $\mathfrak{c}$ and $\text{cov}(\mathcal{M})$. But, assuming that the FS iterations are carried out with length $\omega_1$ in order to get a value of $\mathfrak{c} = \omega_2$, then it is still possible to find a sensible difference between the two FS models and the CS one. Notice that in the models that we got by FS-iterations, we get a strongly summable ultrafilter of character $\omega_1$, and also recall that strongly summable ultrafilters are rapid (i.e. the generators of the ultrafilter form a dominating family), hence we have $\omega_1 = \mathfrak{u} = \mathfrak{d}$. In the CS iteration, on the other hand, we have that $\omega_2 = \mathfrak{u} = \mathfrak{c}$ because each ultraLaver real both adds a dominating real and destroys all ultrafilters from the ground model. So the FS iterations yield different models from the one obtained by means of the CS iteration.

Now, as for comparing the two different FS iterations (the one with Prikry-Mathias forcing and the one that uses ultraLaver), the author has still not been able to find any statement which holds in one but not in the other. Thus, so far we still cannot differentiate those two models, although intuitively they should be very different from each other.

## 4.4 A Strongly Summable Ultrafilter that is not a Union Ultrafilter

Theorem 3.27 from Chapter 3 depends heavily on the hypothesis that the ultrafilter $p$ at hand does not contain the subgroup $B(G) = \{x \in G \big| o(x) = 2\}$, since there are no sequences $\vec{x}$ satisfying the 2-uniqueness of finite sums in $B(G)$. Corollary 3.28 also has that $B(G) \notin p$ as a hypothesis, but it is not entirely clear *a priori* that this hypothesis is necessary for the result. The main objective of this section is to prove that we do in fact need such a hypothesis. This is, if $p \in G^*$ is strongly summable and $B(G) \in p$, then there is no guarantee that $p$ is additively isomorphic to a union ultrafilter. For this, of course, we only need to consider the case where $B(G)$ is infinite (otherwise, the only ultrafilters that can contain it are the principal ones). And, as noted in Chapter 3, when dealing with strongly summable ultrafilters we may assume without loss of generality that $G$ (and hence $B(G)$) is countable. Hence, by focusing our attention on the restricted ultrafilter $p \upharpoonright B(G)$, all we really have to do is work on the Boolean group $\mathbb{B}$.

The rest of this section is devoted to showing that the hypothesis that $\{x \in G \big| o(x) = 2\} \notin p$ in Corollary 3.28 is necessary, by constructing a nonprincipal strongly summable ultrafilter on $\mathbb{B}$ that is not additively isomorphic to a union ultrafilter. This construction borrows lots of ideas from the constructions of un-ordered union ultrafilters that can be found in [6, Th. 4] and [23, Cor. 5.2]. It first appeared in [11, Section 4]. We first show an effective way to look at additive isomorphisms to union ultrafilters.

**Lemma 4.27.** *Let $p \in \mathbb{B}^*$ be a strongly summable ultrafilter that is additively isomorphic to some union ultrafilter. Then there exists a linearly independent $X$ such that $\mathrm{F}\triangle(X) \in p$ and satisfying that whenever $A \subseteq \mathrm{F}\triangle(X)$ is such that $A \in p$, there exists a set $Z$, whose elements have pairwise disjoint $X$-supports, with $p \ni \mathrm{F}\triangle(Z) \subseteq A$.*

*Proof.* If the strongly summable ultrafilter $p \in \mathbb{B}^*$ is additively isomorphic to a union ultrafilter, by Propositions 1.11 and 1.8, we have that for some linearly independent $X$ such that $\mathrm{F}\triangle(X) \in p$ and for some enumeration of $X$ as $X = \{x_n \big| n < \omega\}$, the mapping $\varphi : \mathrm{F}\triangle(X) \longrightarrow [\omega]^{<\omega}$ given by $\bigtriangleup_{n\in a} x_n \longmapsto a$ sends $p$ to a union ultrafilter. Note that the mapping $\varphi$ is a vector space isomorphism from the subspace spanned by $X$, to all of $\mathbb{B}$ (in fact it is the unique linear extension of the mapping $x_n \longmapsto \{n\}$). The fact that $\varphi(p)$ is a union ultrafilter means that, for every $A \subseteq \mathrm{F}\triangle(X)$ such that $A \in p$, there is a pairwise disjoint family $Y$ such that $\varphi(p) \ni \mathrm{F}\triangle(Y) \subseteq \varphi[A]$. Since $Y$ is pairwise disjoint, we get that $\mathrm{F}\triangle(Y) = \mathrm{F}\triangle(Y)$ and since $\varphi$ is an isomorphism, $\varphi^{-1}[\mathrm{F}\triangle(Y)] = \mathrm{F}\triangle(Z)$ where $Z = \varphi^{-1}[Y]$. Now the fact that $Y$ is pairwise disjoint means that the $X$-supports of the elements of $Z$ are pairwise disjoint, and we have that $p \ni \mathrm{F}\triangle(Z) \subseteq A$. $\square$

Thus our goal is to construct, by a transfinite recursion, a strongly summable ultrafilter and somehow, at the same time, for each linearly independent $X$ such that $\mathrm{F}\triangle(X)$ will end up in the ultrafilter, at some stage we need to start making sure that, for every new set of the form $\mathrm{F}\triangle(Z)$ that we are adding to the ultrafilter, the generators $Z$ do not have pairwise disjoint $X$-support. The notions of suitable and adequate families for $X$ will precisely code the way in which we are going to ensure that.

**Definition 4.28.** For a linearly independent subset $X \subseteq G$, we will say that a subset $Y \subseteq \mathrm{F}\triangle(X)$ is **suitable** for $X$ if:

(i) For each $m < \omega$ there exists an $m$-sequence $\langle y_i \big| i < m\rangle$ of elements of $Y$ such that whenever $i < j < m$, the set $\mathrm{supp}_X(y_i) \cap \mathrm{supp}_X(y_j)$ is nonempty. This sequence will be called an **$m$-witness for suitability**.

(ii) Whenever $y, y' \in Y$ are such that $\mathrm{supp}_X(y) \cap \mathrm{supp}_X(y')$ is nonempty, the set $[\mathrm{supp}_X(y) \cap \mathrm{supp}_X(y')] \setminus \mathrm{supp}_X(Y \setminus \{y, y'\})$ is also nonempty. (We do not require here that $y \neq y'$; in particular, for each $y \in Y$, $\mathrm{supp}_X(y) \setminus \mathrm{supp}_X(Y \setminus \{y\})$ is nonempty, and this is easily seen to imply that $Y$ must be linearly independent).

Thus a suitable set $Y$ for $X$ contains, in a carefully controlled way, arbitrarily large bunches of elements whose $X$-supports always pairwise intersect. Given a linearly independent set $X$, it is easy to inductively build a set $Y$ that is suitable for $X$. And once we have such a suitable set, we can look at subsets of $\mathrm{F}\triangle(Y)$ which, in a sense, borrow from $Y$ the non-disjointness of their $X$-supports. This is captured in a precise sense by the following definition, which also captures the fact that we will want to handle the non-disjointness of the $X$-supports for several distinct linearly independent sets $X$ simultaneously.

**Definition 4.29.** Let $A \subseteq \mathbb{B}$ and let $\mathscr{Y} = \{(X_i, Y_i) \big| i < n\}$ be a finite family such that for each $i < n$, $X_i$ is a linearly independent subset of $G$ and $Y_i$ is suitable for $X_i$. Also, let $m < \omega$. Then we will say that $A$ is $(\mathscr{Y}, m)$**-adequate** if there exists an $m$-sequence $\langle a_j \big| j < m\rangle$, called a $(\mathscr{Y}, m)$**-witness for adequacy**, such that for each $i < n$,

(i) $\mathrm{F}\triangle(\vec{a}) \subseteq A \cap \mathrm{F}\triangle(Y_i)$ (which is in turn a subset of $\mathrm{F}\triangle(X_i)$),

(ii) There exists an $m$-witness for the suitability of $Y_i$, $\langle y_j \big| j < m\rangle$, such that for each two distinct $j, k < m$, $y_j \in \mathrm{supp}_{Y_i}(a_j)$ and $y_j \notin \mathrm{supp}_{Y_i}(a_k)$.

If we are given a family of ordered pairs $\mathscr{X}$ all of whose first entries are linearly independent subsets of $\mathbb{B}$, while every second entry is suitable for the corresponding first entry, then we will say that $A$ is $\mathscr{X}$-adequate if it is $(\mathscr{Y}, m)$-adequate for all finite $\mathscr{Y} \subseteq \mathscr{X}$ and for all $m < \omega$. When $\mathscr{Y}$ is a singleton $\{(X, Y)\}$, we will just say that $A$ is $(X, Y)$-adequate.

Requirement (ii) of Definition 4.29 in particular implies that, for $j < k < m$, the set $\operatorname{supp}_{X_i}(a_j) \cap \operatorname{supp}_{X_i}(a_k)$ is nonempty. Thus the $X_i$-supports of the terms of a witness for adequacy are not pairwise disjoint, and moreover their non-disjointness does not happen randomly, but is rather induced by some non-disjointness going on at the level of $Y_i$. Also, note that if $Y$ is suitable for $X$ then $\mathrm{F}\triangle(Y)$ is $(X, Y)$-adequate, with the witnesses for suitability witnessing adequacy at the same time. The following lemma, along with the observation that an $\mathscr{X}$-adequate set is also $(X, Y)$-adequate for each $(X, Y) \in \mathscr{X}$, tells us that this notion of adequacy is adequate (pun intended) for our purpose of banishing sets of the form $\mathrm{F}\triangle(Z)$ for which the elements of $Z$ have pairwise disjoint $X$-supports.

**Lemma 4.30.** *Let $X$ and $Z$ be both linearly independent and let $Y$ be suitable for $X$. Assume that $Z \subseteq \mathrm{F}\triangle(Y)$. If the elements of $Z$ have pairwise disjoint $X$-supports then $\mathrm{F}\triangle(Z)$ is not $(X, Y)$-adequate.*

*Proof.* Clause (ii) from Definition 4.28 implies that, for two distinct $z, z' \in Z$, if $y \in \operatorname{supp}_Y(z)$ and $y' \in \operatorname{supp}_Y(z')$ then $\operatorname{supp}_X(y) \cap \operatorname{supp}_X(y') = \varnothing$, for otherwise $\operatorname{supp}_X(z)$ would not be disjoint from $\operatorname{supp}_X(z')$. Thus $\langle z, z' \rangle$ cannot be an $((X, Y), 2)$-witness. More generally, for any two $w, w' \in \mathrm{F}\triangle(Z)$, the only way that there could exist two distinct $y \in \operatorname{supp}_Y(w)$ and $y' \in \operatorname{supp}_Y(w')$ such that $\operatorname{supp}_X(y) \cap \operatorname{supp}_X(y') \neq \varnothing$ would be if $y, y' \in \operatorname{supp}_Y(z)$ for some $z \in Z$ such that $z \in \operatorname{supp}_Z(w) \cap \operatorname{supp}_Z(w')$. But then $y \in \operatorname{supp}_Y(w')$ and $y' \in \operatorname{supp}_Y(w)$. Hence

$\langle w, w' \rangle$ cannot be an $((X, Y), 2)$-witness and we are done. $\square$

Given this, the idea for the recursive construction of an ultrafilter would be as follows: at each stage we choose some set $\mathrm{F}\triangle(X)$ that has already been added to the ultrafilter, and then we choose a suitable (for $X$) set $Y$. At every stage we make sure that the subsets of $\mathbb{B}$ that we are adding to the ultrafilter are $\mathscr{X}$-adequate, where $\mathscr{X}$ is the collection of all pairs $(X, Y)$ that have been thus chosen so far. If we want to have a hope of succeeding in such a construction, we better make sure that the notion of being $\mathscr{X}$-adequate behaves well with respect to partitions. For this we will need the following lemma.

**Lemma 4.31.** *Let $\mathscr{Y} = \{(X_i, Y_i) \big| i < n\}$ where each $X_i$ is linearly independent and each $Y_i$ is suitable for $X_i$. Let $\vec{a} = \langle a_j \big| j < M \rangle$ be a $(\mathscr{Y}, M)$-witness for adequacy, and let $\langle b_i \big| i < m \rangle$ be an $m$-sequence of pairwise disjoint subsets of $M$. If we define $\vec{c} = \langle c_j \big| j < m \rangle$ by $c_j = \bigtriangleup_{k \in b_j} a_k$, then $\vec{c}$ will be a $(\mathscr{Y}, m)$-witness for adequacy.*

*Proof.* Let us check that $\vec{c}$ satisfies both requirements of Definition 4.29 for a $(\mathscr{Y}, m)$-witness. Fix $i < n$. Since the $b_j$ are pairwise disjoint, we have that $\mathrm{F}\triangle(\vec{c}) \subseteq \mathrm{F}\triangle(\vec{a}) \subseteq A \cap \mathrm{F}\triangle(Y_i)$, thus requirement (i) is satisfied. In order to see that requirement (ii) holds, grab the corresponding $m$-witness for suitability, $\langle y_j \big| j < M \rangle$, as in part (ii) of Definition 4.29 for $\vec{a}$. Now for $j < m$, pick a $k_j \in b_j$ and let $w_j = y_{k_j}$. Since the $w_j$ were chosen from among the $y_k$, the sequence $\vec{w} = \langle w_j | j < m \rangle$ is an $m$-witness for suitability. Now for $j < m$, since $w_j \in \text{supp}_{Y_i}(a_{k_j})$ and $w_j \notin \text{supp}_{Y_i}(a_l)$ for $l \neq k_j$, it follows that $w_j \in \text{supp}_{Y_i}(c_j)$ and $w_j \notin \text{supp}_{Y_i}(c_{j'})$ for $j \neq j'$, and we are done. □

An easy consequence of the previous lemma is the observation that any $(\mathscr{Y}, M)$-adequate set is also $(\mathscr{Y}, m)$-adequate for any $m \leq M$. Lemma 4.31 will allow us to prove the following lemma, which is crucial.

**Lemma 4.32.** *For each $m < \omega$ there is an $M < \omega$ such that whenever $\mathscr{Y}$ is a finite family of ordered pairs of the form $(X, Y)$, with $X$ a linearly independent set and $Y$ suitable for $X$, and whenever a $(\mathscr{Y}, M)$-adequate set is partitioned into two cells, one of the cells must be $(\mathscr{Y}, m)$-adequate.*

*Proof.* For this, we will use a theorem of Graham and Rothschild which is a finitary version of Hindman's theorem, namely: for every $m < \omega$ there is an $M < \omega$ such that whenever we partition $\mathfrak{P}(M) \setminus \{\varnothing\}$ into two cells, then one of the cells contains $\text{F}\triangle(\vec{b})$ for some pairwise disjoint $m$-sequence $\vec{b} = \langle b_i | i < m \rangle$ of nonempty subsets of $M$ (this result is sometimes referred to as the *Folkman-Rado-Saunders* theorem). An elegant proof of this theorem from the infinitary version, using a so-called compactness argument, can be obtained by following the proof of [21, Th. 5.29] as a template, applied to the semigroup whose underlying set is $[\omega]^{<\omega}$ and whose semigroup operation is the union $\cup$.

Thus for $m < \omega$, let $M$ be given by this finitary theorem, and let $A$ be a $(\mathscr{Y}, M)$-adequate set. Let $\vec{a} = \langle a_j \big| j < M\rangle$ be a $(\mathscr{Y}, M)$-witness for the adequacy of $A$. If $A$ is partitioned into the two cells $A_0, A_1$, then since $\mathrm{F}\triangle(a) \subseteq A$, we can induce a partition of $\mathfrak{P}(M) \setminus \{\varnothing\}$ into the two cells $B_0, B_1$ by declaring a subset $s \subseteq M$ to be an element of $B_l$ iff $\bigtriangleup_{j\in s} a_j \in A_l$ for $l \in 2$. Then the theorem of Graham and Rothschild gives us a pairwise disjoint family $\vec{b} = \langle b_j \big| j < m\rangle$ and an $l \in 2$ such that $\mathrm{F}\triangle(\vec{b}) \subseteq B_l$. Letting $\vec{c} = \langle c_j \big| j < m\rangle$ be given by $c_j = \bigtriangleup_{k\in b_j} a_k$, we get that $\mathrm{F}\triangle(\vec{c}) \subseteq A_l$ and Lemma 4.31 ensures that $\vec{c}$ is a $(\mathscr{Y}, m)$-witness for adequacy. Therefore $A_l$ is $(\mathscr{Y}, m)$-adequate and we are done. □

**Corollary 4.33.** *For any family $\mathscr{X}$ consisting of ordered pairs of the form $(X, Y)$, with $X$ a linearly independent set and $Y$ suitable for $X$, if we partition an $\mathscr{X}$-adequate set into two cells, then one of them must be $\mathscr{X}$-adequate.*

*Proof.* If $A = A_0 \cup A_1$ is a partition of the $\mathscr{X}$-adequate set $A$, and neither $A_0$ nor $A_1$ are $\mathscr{X}$-adequate, then the reason for this is the existence of finite $\mathscr{Y}_0, \mathscr{Y}_1 \subseteq \mathscr{X}$ and $m_0, m_1 < \omega$ such that $A_0$ is not $(\mathscr{Y}_0, m_0)$-adequate and $A_1$ is not $(\mathscr{Y}_1, m_1)$-adequate. Pick the $M$ that works for $\max\{m_0, m_1\}$ in Lemma 4.32. Then for some $i \in 2$, $A_i$ is $(\mathscr{Y}_0 \cup \mathscr{Y}_1, \max\{m_0, m_1\})$-adequate (because $A$ is $(\mathscr{Y}_0 \cup \mathscr{Y}_1, M)$-adequate), in particular $A_i$ is $(\mathscr{Y}_i, m_i)$-adequate, a contradiction. □

With these preliminary results under our belt, we are finally ready to prove the main theorem of this section.

**Theorem 4.34.** *If* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*, then there exists a nonprincipal strongly summable ultrafilter on* $\mathbb{B}$ *that is not additively isomorphic to any union ultrafilter (in particular, there is a nonprincipal strongly summable ultrafilter on* $\mathbb{B}$ *that is not a union ultrafilter).*

*Proof.* Let $\{A_\alpha \big| \alpha < \mathfrak{c}\}$ be an enumeration of all subsets of $\mathbb{B}$, and let $\langle X_\alpha \big| \alpha < \mathfrak{c}\rangle$ be an enumeration of all infinite linearly independent subsets of $\mathbb{B}$ in such a way that each such set appears cofinally often in the enumeration. Now recursively define linearly independent sets $\langle Y_\alpha \big| \alpha < \mathfrak{c}\rangle$ and a strictly increasing sequence of ordinals $\langle \gamma_\alpha \big| \alpha < \mathfrak{c}\rangle$ satisfying the following conditions for each $\alpha < \mathfrak{c}$:

(i) $\gamma_\alpha$ is the least $\eta \geq \sup_{\xi<\alpha}(\gamma_\xi + 1)$ such that $\mathrm{F}\triangle(Y_\xi) \subseteq \mathrm{F}\triangle(X_\eta)$ for some $\xi < \alpha$.

(ii) $Y_\alpha$ is suitable for $X_{\gamma_\alpha}$.

(iii) $\mathrm{F}\triangle(Y_\alpha)$ is either contained in or disjoint from $A_\alpha$.

(iv) The family $\mathscr{F}_\alpha = \{\mathrm{F}\triangle(Y_\xi) \big| \xi \leq \alpha\}$ is centred.

(v) Letting $\mathscr{X}_\alpha = \{(X_{\gamma_\xi}, Y_\xi) \big| \xi \leq \alpha\}$, the filter generated by $\mathscr{F}_\alpha$ consists of $\mathscr{X}_\alpha$-adequate sets.

Thus at each stage $\alpha$, we first use clause (i) to determine what $\gamma_\alpha$ will be, and then we work to find a $Y_\alpha$ satisfying (ii)–(v).

Let us first look at what we have at the end of this construction. Clause (iv) tells us that the family $\{\mathrm{F}\triangle(Y_\alpha)\big|\alpha < \mathfrak{c}\}$ generates a filter $p$, which will be an ultrafilter because of (iii), and it will obviously be nonprincipal and strongly summable. Now notice that (v) implies that, if $\mathscr{X}_\mathfrak{c} = \{(X_{\gamma_\alpha}, Y_\alpha)\big|\alpha < \mathfrak{c}\}$, then each $A \in p$ will be $\mathscr{X}_\mathfrak{c}$-adequate, because if $\mathscr{Y} = \{(X_{\gamma_{\alpha_i}}, Y_i)\big|i < n\}$ is a finite subfamily of $\mathscr{X}_\mathfrak{c}$, $m < \omega$, and $A \in p$, then we can grab an $\alpha < \mathfrak{c}$ larger than all $\gamma_{\alpha_i}$ and also larger than the $\beta$ witnessing $\mathrm{F}\triangle(Y_\beta) \subseteq A$. By (v), $\mathrm{F}\triangle(Y_\alpha) \cap \mathrm{F}\triangle(Y_\beta)$ is $\mathscr{X}_\alpha$-adequate, in particular it is $(\mathscr{Y}, m)$-adequate and thus so is $A$.

The last observation is crucial for the argument that $p$ cannot be additively isomorphic to any union ultrafilter. If it was, by Lemma 4.27 there would be a linearly independent $X$ such that $\mathrm{F}\triangle(X) \in p$ and such that for each $A \in p$ satisfying $A \subseteq \mathrm{F}\triangle(X)$, we would be able to find a family $Z$ whose elements have pairwise disjoint $X$-supports and such that $p \ni \mathrm{F}\triangle(Z) \subseteq A$. Now since $\mathrm{F}\triangle(X) \in p$, there is an $\alpha < \mathfrak{c}$ such that $\mathrm{F}\triangle(Y_\alpha) \subseteq \mathrm{F}\triangle(X)$, let $\eta$ be the least ordinal $\geq \sup_{\xi \leq \alpha}(\gamma_\xi + 1)$ such that $X = X_\eta$. By (i) we will have that $\gamma_{\alpha+1} \leq \eta$ and, in fact, whenever $\xi > \alpha$ is such that no $\gamma_\beta$ equals $\eta$ for any $\alpha < \beta < \xi$, then $\gamma_\xi \leq \eta$. Thus there will eventually be some $\zeta > \alpha$ such that $\gamma_\zeta = \eta$, and by (ii) this means that $Y_\zeta$ is suitable for $X$. Since every element of $p$ is $\mathscr{X}_\mathfrak{c}$-adequate, in particular $(X, Y_\zeta)$-adequate, then by Lemma 4.30 we get that for no set $Z$ with pairwise disjoint $X$-supports can we have that $p \ni \mathrm{F}\triangle(Z) \subseteq \mathrm{F}\triangle(Y_\zeta)$. This shows that $p$ cannot be additively isomorphic to any union ultrafilter, and we are done.

We now proceed to show how is it possible to carry out such a construction. So let $\alpha < \mathfrak{c}$ and assume that for all $\xi < \alpha$, conditions (i)–(v) are satisfied. As mentioned before, condition (i) uniquely determines $\gamma_\alpha$, so we only need to focus on constructing $Y_\alpha$ satisfying conditions (ii)–(v). Let $\mathscr{F} = \{\mathrm{F}\triangle(Y_\xi) \big| \xi < \alpha\}$, and $\mathscr{X} = \{(X_{\gamma_\xi}, Y_\xi) \big| \xi < \alpha\}$. Condition (v) implies that the filter generated by $\mathscr{F}$ consists of $\mathscr{X}$-adequate sets, if $\alpha$ is limit, by the same argument as in the proof that $p$ consists of $\mathscr{X}_\mathfrak{c}$-adequate sets, and if $\alpha = \xi + 1$ just because $\mathscr{F} = \mathscr{F}_\xi$ and $\mathscr{X} = \mathscr{X}_\xi$. Thus if we define

$$H = \left\{ q \in \beta\mathbb{B} \middle| (q \supseteq \mathscr{F}) \wedge (\forall A \in q)(A \text{ is } \mathscr{X}-\text{adequate}) \right\},$$

then $H$ will be a nonempty subset of $\beta\mathbb{B}$ by Corollary 4.33. Since finite sets cannot be $\mathscr{X}$-adequate, we have that, in fact, $H \subseteq B^*$.

**Claim 4.2.** *$H$ is a closed subsemigroup of $\mathbb{B}$.*

*Proof of Claim.* The fact that $H$ is closed is fairly straightforward and is left to the reader, and it is also clear (by Theorem 1.7) that $H$ is a subsemigroup.

Now we only need to show that, if $A \in p \blacktriangle q$, then $A$ is $\mathscr{X}$-adequate. So fix a finite $\mathscr{Y} = \{(X_i, Y_i) \big| i < n\} \subseteq \mathscr{X}$ and an $m < \omega$. We will see that there is a

$(\mathscr{Y},m)$-witness for the adequacy of $A$. Let $B=\{x\in\mathbb{B}\big|x\blacktriangle A\in q\}$. We have that $B\in p$ because $A\in p\blacktriangle q$, so $B$ is $\mathscr{X}$-adequate and thus we can grab a $(\mathscr{Y},m)$-witness $\langle a_j\big|j<m\rangle$ for the adequacy of $B$. For each $i<n$, $\mathrm{F}\triangle(\vec{a})\subseteq\mathrm{F}\triangle(Y_i)$ so we can define $Z_i\in[Y_i]^{<\omega}$ by $Z_i=\mathrm{supp}_{Y_i}(\vec{a})$. Consider the set

$$C=\bigcap_{a\in\mathrm{F}\triangle(\vec{a})}a\blacktriangle A,$$

which is an element of $q$ because $\mathrm{F}\triangle(\vec{a})\subseteq B$ and hence it is $\mathscr{X}$-adequate. Therefore we can grab a $(\mathscr{Y},2^{\sum_{i<n}|Z_i|}+2m-1)$-witness for the adequacy of $C$, $\langle b_j\big|j<2^{\sum_{i<n}|Z_i|}+2m-1\rangle$. Associate to any element $x\in\bigcap\limits_{i<n}\mathrm{F}\triangle(Y_i)$ the vector $\langle Z_i\cap\mathrm{supp}_{Y_i}(x)\big|i<n\rangle$, and notice that there are exactly $2^{\sum_{i<n}|Z_i|}$ many possible distinct such vectors. Thus there exist $2m$ distinct numbers $k_0,\dots,k_{2m-1}<2^{\sum_{i<n}|Z_i|}+2m-1$ such that for each $j<m$, the vector associated to $b_{k_{2j}}$ is exactly the same as the one associated to $b_{k_{2j+1}}$, and so if we let $c_j=b_{k_{2j}}\triangle b_{k_{2j+1}}$, then for each $i<n$, $c_j\in\mathrm{F}\triangle(Y_i\setminus Z_i)$. By Lemma 4.31, the $m$-sequence $\vec{c}=\langle c_j\big|j<m\rangle$ will be an $m$-witness for the adequacy of $C$. Now let $\vec{d}=\langle d_j\big|j<m\rangle$ be given by $d_j=a_j\triangle c_j$. We claim that $\vec{d}$ is a $(\mathscr{Y},m)$-witness for the adequacy of $A$, so let us fix $i<n$ and let us verify that $\vec{d}$ satisfies conditions (i) and (ii) from Definition 4.29. It is certainly the case that $\mathrm{F}\triangle(\vec{c})\subseteq A\cap\mathrm{F}\triangle(Y_i)$, because if $d\in\mathrm{F}\triangle(\vec{d})$ then there are $a\in\mathrm{F}\triangle(\vec{a})$ and $c\in\mathrm{F}\triangle(\vec{c})$ such that $d=a\triangle c$, and since $c\in C\subseteq a\blacktriangle A$, we get that $d\in A$. Thus requirement (i) is satisfied. Now for requirement (ii), just grab the $m$-witness for the suitability of $Y_i$ that works for $\vec{a}$, $\langle y_j\big|j<m\rangle$. We constructed

the $c_j$ in such a way that $\operatorname{supp}_{Y_i}(c_j) \cap Z_i = \varnothing$, while $\operatorname{supp}_{Y_i}(a_j) \subseteq Z_i$. Hence for each $j < m$, $\operatorname{supp}_{Y_i}(d_j) \cap Z_i = \operatorname{supp}_{Y_i}(a_j)$ and so whenever $j < m$, $y_j \in \operatorname{supp}_{Y_i}(d_j)$, and $y_j \notin \operatorname{supp}_{Y_i}(d_k)$ for $k \neq j$. □

Since $H$ is a closed subset of the compact space $\beta\mathbb{B}$, then $H$ is compact as well, and since it is a semigroup in its own right, we can apply the Ellis-Numakura Lemma and pick an idempotent element $q \blacktriangle q = q \in H$. Let $A \in \{A_\alpha, \mathbb{B} \setminus A_\alpha\}$ be such that $A \in q$. We will use $q$ to carefully construct $Y_\alpha$. Let $X = X_{\gamma_\alpha}$.

**Claim 4.3.** *There is a $Y$, suitable for $X$, such that:*

*(i)* $\mathrm{F}\triangle(Y) \subseteq A$, *and*

*(ii) For any finite subfamily $\mathscr{Y} = \{(X_i, Y_i) \big| i < n\} \subseteq \mathscr{X}$, for any $m < \omega$ and for any finitely many $\xi_0, \dots, \xi_k < \alpha$, there is a sequence $\langle a_j \big| j < m \rangle$ of elements of $Y$ that is simultaneously an $m$-witness for the suitability (for $X$) of $Y$ and a $(\mathscr{Y}, m)$-witness for the adequacy of $\bigcap_{l \leq k} \mathrm{F}\triangle(Y_{\xi_l})$. In particular, $\vec{a}$ witnesses the $(\mathscr{Y} \cup \{(X, Y)\}, m)$-adequacy of $\left(\bigcap_{l \leq k} \mathrm{F}\triangle(Y_{\xi_l})\right) \cap \mathrm{F}\triangle(Y)$.*

*Proof.* This is the only place where we will actually use the hypothesis that $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$. Since $q$ is an idempotent and $A \in q$, the set $A^\star = \{x \in A \big| x \blacktriangle A \in q\} \in q$ and by [21, Lemma 4.14], for every $x \in A^\star$, $x \blacktriangle A^\star \in q$. Let $\mathbb{P}$ be the partial order consisting of those finite subsets $W \subseteq \mathrm{F}\triangle(X)$ such that $\mathrm{F}\triangle(W) \subseteq A^\star$ and satisfying condition (ii) from the Definition 4.28 of suitability for $X$, ordered by reverse inclusion (thus $Z \leq W$ means that $Z \supseteq W$). This is a countable forcing notion, hence forcing equivalent to Cohen's forcing. For any finite $\mathscr{Y} \subseteq \mathscr{X}$, every $m < \omega$, and all $\xi_0, \ldots, \xi_k < \alpha$ as in part (ii) of the conclusions of this claim, we let $D(\mathscr{Y}, m, \xi_0, \ldots, \xi_k)$ be the set consisting of all conditions $Z \in \mathbb{P}$ such that there is an $m$-sequence $\vec{a}$ of elements of $Z$ that simultaneously witnesses the suitability of $Z$ for $X$ and the $(\mathscr{Y}, m)$-adequacy of $\bigcap_{l \leq k} \mathrm{F}\triangle(Y_{\xi_l})$. The heart of this proof will be the argument that all these sets $D(\mathscr{Y}, m, \xi_0, \ldots, \xi_k)$ are dense in $\mathbb{P}$. Once we have that, we just need to notice that there are $|\alpha| < \mathfrak{c} = \mathrm{cov}(\mathcal{M})$ many such dense sets, so we can pick a filter $G$ intersecting them all, and we will clearly be done by defining $Y = \bigcup G$.

So let us prove that $D(\mathscr{Y}, m, \xi_0, \ldots, \xi_k)$ is dense in $\mathbb{P}$. The idea is that we are given a condition $Z \in \mathbb{P}$, and we would like to pick a $(\mathscr{Y}, m)$-witness $\vec{a}$ for the adequacy of $\bigcap_{l \leq k} \mathrm{F}\triangle(Y_{\xi_l})$, and extend $Z$ to a stronger condition $W$ by adding the range of $\vec{a}$ to it. The main difficulty is that we want $\vec{a}$ to be at the same time an $m$-witness for suitability (for $X$) such that the resulting condition $W = Z \cup \{a_j \big| j < m\}$ still satisfies condition (ii) of Definition 4.28.

Let us start with a condition $Z \in \mathbb{P}$, and let $X' = X \setminus \mathrm{supp}_X(Z)$. Since

$\mathrm{F}\triangle(X') \in q$, we can let

$$B = \left(\bigcap_{l \leq k} \mathrm{F}\triangle(Y_{\xi_l})\right) \cap \mathrm{F}\triangle(X') \cap \left(\bigcap_{z \in \mathrm{F}\triangle(Z)} z \blacktriangle A^\star\right).$$

Then $B^\star = \{x \in B \big| x \blacktriangle B \in q\} \in q$, thus $B^\star$ is $\mathscr{X}$-adequate, so there is a $(\mathscr{Y}, m)$-witness $\vec{a} = \langle a_j \big| j < m\rangle$ for the adequacy of $B^\star$. We will now recursively construct an $m + \binom{m}{2}$-sequence of elements $\vec{x} = \langle x_k \big| k < m + \binom{m}{2}\rangle$ such that $\mathrm{F}\triangle(\vec{x}) \subseteq \bigcap_{a \in \mathrm{F}\triangle(\vec{a})} a \blacktriangle B^\star$ and such that the $X$-supports of its elements are pairwise disjoint and also disjoint from $\mathrm{supp}_X(\vec{a})$, and whose $Y_i$-supports are disjoint from $\mathrm{supp}_{Y_i}(\vec{a})$ for each $i < n$. If we succeed in this construction, picking a bijection $f : [m]^2 \longrightarrow (m + \binom{m}{2}) \setminus m$ will enable us to define the sequence $\vec{b} = \langle b_j \big| j < m\rangle$ by:

$$b_j = a_j \bigtriangleup x_j \bigtriangleup \left(\mathop{\Large\bigtriangleup}_{\substack{k<m \\ k \neq j}} x_{f(\{j,k\})}\right).$$

Since the $Y_i$-supports of all the $x_k$ are disjoint from $\mathrm{supp}_{Y_i}(\vec{a})$, then arguing as in the proof of Claim 4.2 we conclude that $\vec{b}$ is a $(\mathscr{Y}, m)$-witness for the adequacy of $B^\star$, hence also for the adequacy of $\bigcap_{l \leq k} \mathrm{F}\triangle(Y_{\xi_l})$. And the careful choice of the $X$-supports of the $x_k$ ensures that $\vec{b}$ is at the same time an $m$-witness for suitability for $X$, hence letting $W = Z \cup \{b_j \big| j < m\}$ yields a condition in $\mathbb{P}$ (i.e. $W$ satisfies condition (ii) of Definition 4.28).

Thus, the only remaining issue is that of picking the $x_k$. Assume that we have picked $x_l$ for $l < k$, and we will show how to pick $x_k$. Since $q$ is an idempotent and

$$C = \bigcap_{a \in \mathrm{F}\triangle(\vec{a} \frown \langle x_l \big| l<k \rangle)} a \blacktriangle B^\star \in q,$$

then $C$ is an IP-set, so there is a linearly independent family $V$ such that $\mathrm{F}\triangle(V) \subseteq C$. As in the argument for the proof of Claim 4.2, to each element $x \in C$ we associate the vector

$$\langle \mathrm{supp}_{Y_i}(\vec{a}) \cap \mathrm{supp}_{Y_i}(x) \big| i < n \rangle \frown \langle \mathrm{supp}_X(\{a_j \big| j < m\} \cup \{x_l \big| l < k\}) \cap \mathrm{supp}_X(x) \rangle,$$

and notice that, since there are only finitely many possible distinct such vectors, the infinite set $V$ must contain at least one pair of distinct elements $v, w$ that have the same associated vector. Hence by letting $x_k = v \triangle w \in \mathrm{F}\triangle(V) \subseteq C$, we get that $\mathrm{supp}_{Y_i}(x_k) \cap \mathrm{supp}_{Y_i}(\vec{a}) = \varnothing$ for all $i < n$, and $\mathrm{supp}_X(x_k) \cap \mathrm{supp}_X(\{a_j \big| j < m\} \cup \{x_l \big| l < k\}) = \varnothing$, so the construction can go on and we are done. □

Let $Y_\alpha = Y$. Obviously requirement (ii) is satisfied, and since $\mathrm{F}\triangle(Y_\alpha) \subseteq A \in \{A_\alpha, \mathbb{B} \setminus A_\alpha\}$, requirement (iii) is satisfied as well. It is easy to see that condition (ii) from the conclusion of the claim ensures at once that requirements (iv) and (v) are fulfilled, and we are done. □

# Bibliography


[1] Tomek Bartoszyński and Haim Judah. *Set Theory. On the Structure of the Real Line.* A. K. Peters, Wellesley, Massachusetts, 1995.

[2] James Baumgartner. A Short Proof of Hindman's Theorem. *J. Combin. Theory Ser. A.*, (17):384–386, 1974.

[3] Andreas Blass. Ultrafilters related to Hindman's finite-unions theorem and its extensions. In S. Simpson, editor, *Logic and Combinatorics*, volume 81, pages 89–124. Contemp. Math. 65(1987), 1987.

[4] Andreas Blass. Homogeneous sets from several ultrafilters. *Topology Appl.*, (156):25812594, 2009.

[5] Andreas Blass. Combinatorial Cardinal Characteristics of the Continuum. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*, volume 1, pages 395–489. Springer, 2010.

[6] Andreas Blass and Neil Hindman. On Strongly Summable Ultrafilters and Union Ultrafilters. *Trans. Amer. Math. Soc.*, (304):83–99, 1987.

[7] Jörg Brendle. Strolling through Paradise. *Fund. Math.*, (148):1–25, 1995.

[8] David Chodounský. *On the Katowice Problem.* PhD thesis, Charles University in Prague, 2011.

[9] Todd Eisworth. Near coherence and filter games. *Arch. Math. Logic*, (40):235–242, 2001.

[10] Todd Eisworth. Forcing and stable-ordered union ultrafilters. *J. Symbolic Logic*, (67):449–464, 2002.

[11] David Fernández Bretón. Strongly Summable Ultrafilters, Union Ultrafilters, and the Trivial Sums Property. arXiv 1306.5421 (submitted).

[12] David Fernández Bretón. Every Strongly Summable Ultrafilter on $\bigoplus \mathbb{Z}_2$ is Sparse. *New York J. Math.*, (19):117–129, 2013.

[13] David Fernández Bretón and Martino Lupini. Strongly productive ultrafilters on semigroups. arXiv 1309.3636 (submitted).

[14] Martin Goldstern, Jakob Kellner, Saharon Shelah, and Wolfgang Wohofsky. Borel conjecture and dual Borel conjecture. *Trans. Amer. Math. Soc.*, (366):245–307, 2014.

[15] Neil Hindman. The existence of certain ultrafilters on $\mathbb{N}$ and a conjecture of Graham and Rothschild. *Proc. Amer. Math. Soc.*, (36):341–346, 1972.

[16] Neil Hindman. Finite Sums from Sequences Within Cells of a Partition of $N$. *J. Combin. Theory Ser. A.*, (17):1–11, 1974.

[17] Neil Hindman. Summable ultrafilters and finite sums. In S. Simpson, editor, *Logic and Combinatorics*, pages 263–274. Contemp. Math. 65(1987), 1987.

[18] Neil Hindman and Lakeshia Legette Jones. Idempotents in $\beta S$ that are only products trivially. *New York J. Math.*, (20):57–80, 2014.

[19] Neil Hindman, Igor Protasov, and Dona Strauss. Strongly Summable Ultrafilters on Abelian Groups. *Matematychni Studii*, (10):121–132, 1998.

[20] Neil Hindman, Juris Steprāns, and Dona Strauss. Semigroups in which all strongly summable ultrafilters are sparse. *New York J. Math.*, (18):835–848, 2012.

[21] Neil Hindman and Dona Strauss. *Algebra in the Stone-Čech Compactification*. Walter de Gruyter, Berlin, second edition, 2012.

[22] Peter Krautzberger. On strongly summable ultrafilters. *New York J. Math.*, (16):629–649, 2010.

[23] Peter Krautzberger. On Union Ultrafilters. *Order*, (29):317–343, 2012.

[24] Peter Krautzberger. On rapid idempotent ultrafilters. *Semigroup Forum*, 2014.

[25] Kenneth Kunen. Some points in $\beta N$. *Math. Proc. Cambridge Philos. Soc.*, (80):385–398, 1976.

[26] Kenneth Kunen. *Set Theory. An introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North Holland, 1980.

[27] Lakeshia Legette. Maximal groups in $\beta S$ can be trivial. *Topology Appl.*, (156):2632–2641, 2009.

[28] Maryanthe Malliaris and Saharon Shelah. Cofinality spectrum theorems in model theory, set theory, and general topology. arXiv 1208.5424.

[29] Maryanthe Malliaris and Saharon Shelah. General topology meets model theory, on $\mathfrak{p}$ and $\mathfrak{t}$. *Proc. Natl. Acad. Sci. USA*, (110):13300–13305, 2013.

[30] V. I. Malyhin. Extremally Disconnected and Similar Groups. *Dokl. Math. (Translation of the mathematics section of Dokl. Akad. Nauk)*, (16):21–25, 1975.

[31] Pierre Matet. Some Filters of Partitions. *J. Symbolic Logic*, (53):540–553, 1988.

[32] Arnold Miller. There are no $Q$-points in Laver's model for the Borel conjecture. *Proc. Amer. Math. Soc.*, (78):103–106, 1980.

[33] Igor Protasov. Finite groups in $\beta G$. *Matematychni Studii*, (10):17–22, 1998.

[34] Saharon Shelah. *Proper and Improper Forcing.* Perspectives in Mathematical Logic. Springer, 1998.

[35] Wolfgang Wohofsky. On the existence of p-points and other ultrafilters in the Stone-Čech-compactification of $\mathbb{N}$. Diploma thesis, Vienna Institute of Technology, 2008.