

HINDMAN AND OWINGS-LIKE THEOREMS WITHOUT THE AXIOM OF CHOICE

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ABSTRACT. We investigate Hindman- and Owings-type Ramsey-theoretic statements in Zermelo–Fraenkel set theory without the Axiom of Choice, with some occasional extra assumptions (such as the Axiom of Dependent Choice and/or the Axiom of Determinacy). We study several variations of Hindman’s theorem on \mathbb{Q} -vector spaces; notably, we show that the uncountable analog of Hindman’s theorem fails for the additive group of \mathbb{R} (under **ZF**), and for \mathbb{Q} -vector spaces of uncountable dimension (under **DC** if such dimension is not well-orderable), among other results. In contrast, for Owings-type configurations, we obtain several positive results, especially when assuming **AD**. These results highlight the interaction between determinacy, algebraic structure, and dimension in the study of infinite Ramsey theory without the Axiom of Choice.

1. INTRODUCTION

Ramsey theory studies the emergence of structured monochromatic configurations inside arbitrary colourings of large sets. Beginning with Ramsey’s theorem [19], a broad class of results can be expressed via partition relations, and the modern viewpoint emphasizes how set-theoretic strength and algebraic structure interact in infinite combinatorics (see, e.g., [11, 12, 15]).

In additive Ramsey theory, Hindman’s theorem [7] is a central example: every finite colouring of \mathbb{N} admits an infinite set X such that the set of all finite sums $\text{FS}(X)$ is monochromatic. This phenomenon connects naturally with algebra in the Stone–Čech compactification and related structural methods [10]. A major theme in the last decade has been the extent to which Hindman-type statements persist beyond the countable setting. Fernández-Bretón showed that Hindman’s theorem is essentially countable, in the sense that uncountable analogues fail on any commutative cancellative semigroup [2], and further strong failures for higher analogues were established in [4]. More recently, Hindman-like principles with uncountably many colours and finite monochromatic configurations have been investigated in [3, 13].

A second, closely related line of work concerns *pairwise-sum* (Owings-type) configurations. Owings [18] asked whether every finite colouring of \mathbb{N} contains an infinite set X such that $X + X$ is monochromatic. Despite some early partial progress [8], the problem remains open to this day; versions of the same problem for \mathbb{R} instead of \mathbb{N} have been studied extensively: see for instance [9, 14, 16] and the development of methods aimed at avoiding large-cardinal assumptions [20].

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In this paper, we revisit both Hindman- and Owings-type statements in ZF, without assuming any choice-related principle when possible, but utilizing hypotheses such as the Axiom of Dependent Choice DC or the Axiom of Determinacy AD when necessary. The original motivation for this work was to study these Ramsey-theoretic principles under AD, or under AD + DC if necessary, but we realized along the way that several of the proofs can actually be carried out in ZF or in ZF + DC only. This will be the criterion that informs the particular choice of additional assumptions beyond ZF. Hence, **all of the proofs in this paper take place in ZF, unless an extra assumption is explicitly noted in the header of the corresponding theorem.**

Main contributions. Our results fall into two families, reflecting the distinct nature of finite-sum and pairwise-sum configurations. Throughout the remainder of the paper, we will refer to a “ κ - θ configuration” to denote the relevant Ramsey-type statement for θ colours, where one attempts to obtain a monochromatic set generated by κ elements.

- **Hindman-like statements.** We begin by showing positive results for both the finite-finite and infinite-finite configurations on any Abelian group; this largely follows directly from the original Hindman’s theorem, and only ZF is needed. In contrast, we obtain negative results for the uncountable-finite configuration, both in \mathbb{R} and in any \mathbb{Q} -vector space with basis (the former in ZF, the latter in ZF + DC); this highlights the role of cancellativity and the rigid support structure of such spaces (cf. [2]). The result for \mathbb{R} in ZF is especially revealing, because it was formerly known only as a corollary of [2, Theorem 5], requiring the use of the Axiom of Choice. Along the way, we also show another negative result for \mathbb{R} regarding the finite-infinite configuration; while this result constitutes a particular case of [3, Theorem 12] in ZFC, it was not known before as a result in ZF only. Finally, we obtain a negative result for the 3-infinite configuration on any \mathbb{Q} -vector space with basis, generalizing [3, Theorem 5]; on the other hand, determinacy-driven partition properties at ω_1 and ω_2 yield a positive statement for the 2-infinite configuration, under AD, whenever we are dealing with a \mathbb{Q} -vector space with a basis of cardinality ω_1 or ω_2 .
- **Owings-like statements.** Under AD, we obtain a positive result for the infinite-infinite configuration in \mathbb{R} , placing Owings-type configurations in a determinacy setting and exploiting regularity phenomena available under this axiom. In contrast, we obtain a negative result for the finite-infinite configuration in \mathbb{Q} -vector spaces with basis. Finally, regarding the infinite-finite configuration (the configuration where most questions remain open, even in the ZFC context), we are able to obtain a positive result under AD for \mathbb{Q} -vector spaces of dimension ω_1 or ω_2 , in the spirit of [16].

Organization of the paper. Section 2 develops Hindman-like results: we introduce the finite-sums arrow notation, establish the simplest cases that follow directly from the classical Hindman’s theorem, and set up the structural lemmas used throughout; after that we prove the main negative and positive theorems for \mathbb{Q} -vector spaces. Section 3 treats Owings-like results: we formalize the pairwise-sum arrow notation, prove the determinacy-based theorem for \mathbb{R} under countable colourings, discuss limitations in cancellative groups, and conclude with the positive

theorem for \mathbb{Q} -vector spaces of dimension ω_i ($i \in \{1, 2\}$) for finitely many colours. There is a Section 4 where we summarize the results obtained, the additional axioms needed for each, and propose further directions of study.

2. HINDMAN-LIKE RESULTS

In this section we analyze finite-sums partition relations in Abelian groups, with particular emphasis on the additive group of \mathbb{R} and other \mathbb{Q} -vector spaces. We first introduce the relevant notation, briefly discuss the simplest cases (the finite-finite and infinite-finite configurations), and set up a combinatorial tool that will allow us to control supports of vectors. We then prove that uncountable finite-sums homogeneity fails in \mathbb{R} (in ZF), and also in vector spaces of uncountable dimension (with DC as an additional assumption). There is also a negative result for the 3-infinite configuration on \mathbb{Q} -vector spaces with basis. Finally, we contrast this negative phenomenon with positive finite configurations obtained from strong partition properties at ω_1 and ω_2 under AD.

2.1. Finite-sums notation and combinatorial tools. As is customary in infinite combinatorics, this text adopts Hungarian notation. Recall that, if X is a subset of some additive structure, $\text{FS}(X)$ denotes the **set of finite sums from X** , given by $\text{FS}(X) = \{\sum_{a \in F} a \mid F \subseteq X \text{ is finite and nonempty}\}$.

Definition 2.1. Let $(G, +)$ be a commutative cancellative semigroup, and let κ, θ be cardinals. We write $G \rightarrow (\kappa)_\theta^{\text{FS}}$ to denote the statement that for every colouring of G with θ colours, there exists a set $M \subseteq G$ of cardinality κ such that $\text{FS}(M)$ is monochromatic.

Hindman's theorem [7] is the statement that $\mathbb{N} \rightarrow (\omega)_\theta^{\text{FS}}$ for all finite θ . Another related statement (known to be equivalent to Hindman's theorem even before it was proved) is that if $[\omega]^{<\omega} \setminus \{\emptyset\}$ is finitely coloured, there is an infinite pairwise disjoint family B such that $\text{FU}(B)$ is monochromatic, where $\text{FU}(B)$ denotes the set of unions of finitely many elements from B . Both of the statements just mentioned can be proven in ZF. From here, one can use a compactness argument to obtain finitary versions of the above statements: given finite θ and κ , there exists some natural number F such that, for every colouring of $\wp(F) \setminus \{\emptyset\}$ in θ colours, there is a pairwise disjoint family $B \subseteq \wp(F) \setminus \{\emptyset\}$ of size κ such that $\text{FU}(B)$ is monochromatic. Similarly (or by mapping each finite $a \subseteq \mathbb{N}$ to the number $\sum_{i \in a} 2^i$), for each finite θ and κ , there is a number R (it suffices to take $R = 2^F - 1$ where F is as in the previous statement) such that, for every colouring of $\{1, \dots, R\}$ in θ colours, there is $M \subseteq R$ of size κ such that $\text{FS}(M)$ is monochromatic.

We generalize the previous results to all Abelian groups, thus settling the finite-finite and infinite-finite configurations for Hindman's theorem.

Remark 2.2. Theorems 2.3 and 2.4 can be stated for all commutative cancellative semigroups, since every commutative cancellative semigroup embeds into a group. We officially state them for Abelian groups for the sake of simplicity.

Theorem 2.3. *For all finite κ, θ there exists an $S \in \mathbb{N}$ such that, for every Abelian group G with $|G| \geq S$, $G \rightarrow (\kappa)_\theta^{\text{FS}}$.*

Proof. Given finite κ and θ , take the corresponding numbers F, R as described in the previous paragraph, and let $S := \max\{R + 1, R^F + 1\}$. Then, if G is an abelian group with $|G| \geq S$ and $c : G \rightarrow \theta$, the following two cases are exhaustive:

$\exists g \in G : o(g) > R$: In this case, the set $\{g, 2g, \dots, Rg\}$ is in bijection with $\{1, 2, \dots, R\}$; furthermore, this bijection preserves addition whenever the result of a sum stays within the set. Hence it suffices to let the restriction $c \upharpoonright \{g, 2g, \dots, Rg\}$ induce a corresponding colouring in $\{1, \dots, R\}$ and apply the finite version of Hindman's theorem.

$\forall g \in G : o(g) \leq R$: Then we recursively choose $h_n \in G \setminus \langle \{h_i | i < n\} \rangle$; the assumption implies that $|\langle \{h_i | i < n\} \rangle| \leq R^n$ and therefore there is always at least one such h_n whenever $n \leq F$. By considering the map $d : \wp(S) \setminus \{\emptyset\} \rightarrow G$ given by $d(f) = \sum_{n \in f} h_n$ and applying the finite version of the finite-unions theorem to $c \circ d : \wp(S) \rightarrow \theta$, there is a disjoint family $B = \{b_i | i < \kappa\} \subseteq \wp(F) \setminus \{\emptyset\}$ such that $\text{FU}(B)$ is $c \circ d$ -monochromatic. This readily implies that, if we let $M = \{d(b_n) | n < \kappa\}$, $\text{FS}(M)$ is c -monochromatic. \square

By essentially the same argument, we are able to obtain the version of the theorem where one obtains an infinite monochromatic set.

Theorem 2.4. *Let $(G, +)$ be a Dedekind-infinite Abelian group. Then, for every finite θ , we have that $G \rightarrow (\omega)_\theta^{\text{FS}}$.*

Proof. Given $c : G \rightarrow \theta$, we have two cases:

$\exists g \in G : o(g) = \infty$: In this case, there is a semigroup isomorphism between $\{ng | n \in \mathbb{N}\}$ and \mathbb{N} , so it suffices to apply Hindman's finite-sums theorem.

$\forall g \in G : o(g) < \infty$: Since G is Dedekind infinite, fix an injection $h : \omega \rightarrow G$ and recursively define $i_n := \min(h[\omega] \setminus \langle \{h(j) | j < n\} \rangle)$; Then, the function $d : [\omega]^{<\omega} \setminus \{\emptyset\} \rightarrow G$ defined by $d(f) = \sum_{n \in f} i_n$ maps every finite union of disjoint sets into the finite sum of the corresponding elements. Hence, by the finite-unions version of Hindman's theorem, there is a pairwise disjoint family $B = \{b_n | n < \omega\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that $\text{FU}(B)$ is d -monochromatic; setting $M = \{d(b_n) | n < \omega\}$, we must have that $\text{FS}(M)$ is c -monochromatic. \square

Remark 2.5. As a corollary of Theorem 2.4, we obtain the following statement: assuming that every infinite set is Dedekind-infinite, for every infinite Abelian group G and for every finite θ we have $G \rightarrow (\omega)_\theta^{\text{FS}}$. In particular, the conclusion follows from each of the axioms of Countable Choice and Dependent Choice.

It is of note that the previous proof requires the assumption that G is Dedekind infinite, as without it, the result may fail, even for infinite Boolean groups, as shown in [6, Theorem 5.2].

The remainder of this subsection is devoted to the study of Δ -systems and the Δ -system lemma, a combinatorial device that will be crucial in what follows.

Definition 2.6. A collection of sets D is a Δ -system if there exists a set R (called the **root** of the Δ -system) such that $c \cap d = R$ for any two distinct $c, d \in D$.

A stronger version of the following lemma is well-known in the ZFC context [1]; here we make explicit a weaker version that only requires DC.

Lemma 2.7 (DC). *Let C be an uncountable collection of finite sets. Then there exists an infinite Δ -system $D \subseteq C$.*

Proof. Since DC implies (the Axiom of Countable Choice, which in turn implies) that countable unions of countable sets are countable, we deduce that there must be an $n \in \mathbb{N}$ such that uncountably many elements of C have cardinality n . So we may assume without loss of generality that every element of C has the same cardinality n , and from here one can develop the usual proof of the lemma by induction on n ; we sketch the proof for the benefit of the reader, and only explain the full details in the parts where the use of DC needs to be made explicit.

The base of the induction, i.e., the case $n = 1$, is immediate (in this case C is already a pairwise disjoint family, that is, a Δ -system with empty root). Now, if we assume $n > 1$, there are two cases, the easy one being if there exists an r such that $r \in c$ for uncountably many $c \in C$ (in this case, apply the induction hypothesis to obtain an infinite Δ -system $D' \subseteq \{c \setminus \{r\} \mid r \in c \in C\}$, and simply let $D = \{d \cup \{r\} \mid d \in D'\}$). The more involved case is when, for every r , there are at most countably many $c \in C$ with $r \in c$. This readily implies that, for every finite (or even countably infinite) subset $F \subseteq C$, all but countably many elements of C are disjoint with every element of F (in case F is countably infinite, one would use again DC to support this claim). Hence, using DC, one can recursively find an infinite sequence of elements $c_m \in C$ with each c_m disjoint from all of the c_k for $k < m$, so that $D = \{c_m \mid m < \omega\} \subseteq C$ is an infinite Δ -system (with empty root). \square

The importance of Lemma 2.7 lies in its facilitation of several structural arguments over \mathbb{Q} -vector spaces. The attentive reader will note that, in case C is assumed to be a collection of subsets of some ordinal number, one does not need the use of DC (since one is able to always choose minimums when necessary) and, in fact, the usual Δ -system lemma, where one obtains an uncountable Δ -system, holds in this context. However, we will be concerned primarily with the case where the elements of C are fully arbitrary.

2.2. Negative results. We now show some failures of Hindman-type statements, especially regarding the uncountable-finite configuration. In the ZFC context, these failures were proved in [2]; with some much stronger results established in [4]. Similar results for the infinite-infinite configuration were established in [3, Theorem 12]. The first surprising result is that, for the additive group of \mathbb{R} , the same negative statements can be proved in ZF.

Theorem 2.8. $\mathbb{R} \not\rightarrow (2)_\omega^{\text{FS}}$.

Proof. Let $c : \mathbb{R} \rightarrow \mathbb{Z}$ be given by

$$c(r) = \begin{cases} k & \text{if } r \in [2^k, 2^{k+1}), \\ 0 & \text{if } r = 0, \\ -k & \text{if } r \in (-2^{k+1}, -2^k]. \end{cases}$$

Suppose that $r, s \in \mathbb{R}$ are distinct and $c(r) = c(s)$. We will show that $c(r+s) \neq c(r)$. One may assume without loss of generality that $r, s > 0$. Set $c(r) = c(s) = k \in \mathbb{Z}$, then $2^k \leq r, s < 2^{k+1}$, which implies $2^{k+1} \leq r+s < 2^{k+2}$. Hence $c(r+s) = k+1$. \square

As a corollary of the previous result, we obtain the following negative result for uncountable monochromatic sets over \mathbb{R} ; a particular case of [2, Theorem 5] in ZFC, but somewhat surprising in the ZF context.

Corollary 2.9. $\mathbb{R} \not\rightarrow (\text{uncountable})_2^{\text{FS}}$.

Proof. Consider the same colouring c as in the previous theorem and, given $r \in \mathbb{R}$, let $f(r)$ be the absolute value of $c(r)$, that is, $f(r) = k$ if $|r| \in [2^k, 2^{k+1})$ and $f(0) = 0$. We now define the colouring $d : \mathbb{R} \rightarrow 2$ by letting $d(r) \equiv f(r) \pmod{2}$. Suppose that $X \subseteq \mathbb{R}$ is an uncountable set, whose elements have the same colour under d . Since X is uncountable, the function c cannot be one-to-one on X , so there are two distinct $r, s \in X$ such that $c(r) = c(s)$. Now, reasoning as in the previous theorem, we conclude $f(r+s) = f(r) + 1$, and so $d(r+s) \neq d(r)$. \square

We now proceed to prove a seemingly more general version of the previous result; bear in mind, however, that we add the extra assumption of the existence of a basis (in ZF, one cannot guarantee that \mathbb{R} has a basis as a \mathbb{Q} -vector space). The proof is an adaptation of [2, Theorem 5] to the ZF context; the key to the argument lies in the algebraic structure of \mathbb{Q} -vector spaces, as their cancellation properties prevent the stabilization phenomena required for uncountable finite-sums configurations.

Theorem 2.10 (DC). *If V is a \mathbb{Q} -vector space with basis, then $V \rightarrow (\text{uncountable})_2^{\text{FS}}$.*

Proof. Let \mathcal{B} be a basis of V , so that $V \cong \bigoplus_{b \in \mathcal{B}} \mathbb{Q}$. Let us define $\text{supp} : V \rightarrow [\mathcal{B}]^{<\omega}$ as follows: given $v \in V$, write $v = \sum_{b \in \mathcal{B}} q_b b$ and let $\text{supp}(v) = \{b \in \mathcal{B} \mid q_b \neq 0\}$. Then the colouring $c : V \rightarrow 2$ is defined by letting $c(v) \equiv \lfloor \log_2 |\text{supp}(v)| \rfloor \pmod{2}$. Proceeding by contradiction, suppose $X \subseteq V$ is an uncountable set such that $\text{FS}(X)$ is monochromatic, say in colour i . Since we are assuming DC, and therefore countable unions of countable sets are countable, there must be an $N \in \mathbb{N}$ such that for uncountably many $v \in X$ we have $|\text{supp}(v)| = N$, so assume without loss of generality that this holds of all $v \in V$. Similarly, letting for a $v = \sum_{b \in \mathcal{B}} q_b b \in V$, $\text{coef}(v) = \{q_b \mid b \in \mathcal{B}\} \setminus \{0\}$, an analogous reasoning allows us to assume, without loss of generality, that there is a fixed $F \in [\mathbb{Q} \setminus \{0\}]^{\leq N}$ such that $(\forall v \in X)(\text{coef}(v) = F)$. Once again we use that countable unions of countable sets are countable to notice that $\{\text{supp}(v) \mid v \in X\}$ is an uncountable set, so we may use Lemma 2.7 together with DC to get a sequence $\{v_n \mid n < \omega\} \subseteq X$ with pairwise distinct supports such that $\{\text{supp}(v_n) \mid n < \omega\}$ forms a Δ -system, say with root R . Since there are at most $|R|^{|F|}$ (a finite number) coefficients for the v_n in the positions of the root R , by the pigeonhole principle, we may assume that the coefficients of the v_n in R are constant, guaranteeing that when adding vectors of W the terms corresponding to R do not cancel. This implies that, if $v_{n_1}, \dots, v_{n_L} \in X$ are distinct, then $\left| \text{supp} \left(\sum_{i=1}^L v_{n_i} \right) \right| = |R| + L(N - |R|)$. So the cardinalities of the supports of elements of $\text{FS}(X)$ form an infinite arithmetic progression (with base N and difference $N - |R|$); letting m be such that $2^m \leq N < 2^{m+1}$ and $n \leq m$ be such that $2^n \leq N - |R| < 2^{n+1}$, we see that

$$2^{m+1} = 2^{m-n} 2^n + 2^m \leq 2^{m-n} (N - |R|) + N \leq 2^{m-n} 2^{n+1} + 2^{m+1} = 2^{m+2},$$

so that any finite sum of $2^{m-n} + 1$ elements from V must have a support with cardinality $|R| + (2^{m-n} + 1)(N - |R|) = 2^{m-n}(N - |R|) + N$. So if $v \in X$ is arbitrary, and $w \in \text{FS}(X)$ is any sum with $2^{m-n} + 1$ summands, we must have $c(v) \equiv m \pmod{2}$ and $c(w) \equiv m + 1 \pmod{2}$, contradicting that $\text{FS}(X)$ is monochromatic. \square

The preceding theorems show that the finite-sums phenomenon remains essentially countable in many \mathbb{Q} -vector spaces.

Moving on to the finite-infinite configuration, we record an additional finite failure which further illustrates the rigidity imposed by cancellation. A version of the following result was established in [3, Theorem 5] in the ZFC context; the proof below does not use AC.

Theorem 2.11. *Let V be a \mathbb{Q} -vector space with basis. Then $V \not\rightarrow (3)_\omega^{\text{FS}}$.*

Proof. Given a basis \mathcal{B} for V , we define the function $\langle | \rangle : V \times V \rightarrow \mathbb{Q}$ by $\langle v|w \rangle = \sum_{b \in \mathcal{B}} x_b y_b$, where $v = \sum_{b \in \mathcal{B}} x_b b$ and $w = \sum_{b \in \mathcal{B}} y_b b$; all of these sums are well-defined because $x_b = y_b = 0$ for all but finitely many $b \in \mathcal{B}$. It is hard not to see that the function just defined satisfies all the properties of an inner product; we now proceed to define the colouring $c : V \rightarrow \mathbb{Q}$ by $c(v) = \langle v|v \rangle$. Suppose that there are three distinct vectors $u, v, w \in V$ such that $\text{FS}(\{u, v, w\})$ is monochromatic, say in colour $q \in \mathbb{Q}$. Then we must have $q \neq 0$ and, in addition,

$$q = \langle u|u \rangle = \langle u + v|u + v \rangle = \langle u|u \rangle + \langle v|v \rangle + 2\langle u|v \rangle,$$

from where one can deduce that $\langle u|v \rangle = -\frac{1}{2}q$. In an entirely similar manner one shows that $\langle v|w \rangle = \langle u|w \rangle = -\frac{1}{2}q$, and therefore

$$\begin{aligned} \langle u + v + w|u + v + w \rangle &= \langle u|u \rangle + \langle v|v \rangle + \langle w|w \rangle + 2\left(\langle u|v \rangle + \langle v|w \rangle + \langle u|w \rangle\right) \\ &= 3q + 2\left(-\frac{3}{2}q\right) \\ &= 0, \end{aligned}$$

meaning in particular that $c(u + v + w) = 0 \neq q = c(u)$, a contradiction. \square

2.3. Positive finite configurations from partition properties. Despite the negative results above, strong partition properties available under determinacy yield positive finite-sums configurations in sufficiently large dimensions; although for this we do need to assume AD. The proof below, reproduced here for the benefit of the reader, is a classic argument made many times over to deduce some small monochromatic finite-sums configurations from Ramsey-type properties, using the fact that AD ensures that ω_1, ω_2 have large-cardinal properties.

Definition 2.12. Let λ, κ, ι and θ be cardinals. We write $\lambda \rightarrow (\kappa)_\theta^\iota$ for the statement: “for every colouring of $[\lambda]^\iota$ into θ colours, there is a monochromatic subset of size κ ,” i.e.

$$(\forall c : [\lambda]^\iota \rightarrow \theta) (\exists M \in [\lambda]^\kappa) (|c[[M]^\iota]| = 1).$$

Theorem 2.13. *Every measurable cardinal κ is weakly compact [11, Lemma 10.18] and thus satisfies $\kappa \rightarrow (\kappa)_\theta^\iota$ for every $\iota < \omega$ and every $\theta < \kappa$ [12, Theorem 7.8].*

In order to reinforce the idea that AD provides strong Ramsey-type consequences, we recall the following classical results.

Theorem 2.14 (Martin [12, Theorem 28.2]). *Under ZF + AD, ω_1 is measurable. Moreover, $\omega_1 \rightarrow (\omega_1)_2^{\omega_1}$; hence $\omega_1 \rightarrow (\omega_1)_\theta^{\omega_1}$ for every $\theta < \omega_1$.*

Theorem 2.15 (Solovay [12, Theorem 28.6]). *Under ZF + AD, ω_2 is measurable.*

Theorem 2.16 (AD). *Let V be a \mathbb{Q} -vector space of well-orderable dimension.*

- (1) *If $\dim(V) = \omega_1$, then $V \rightarrow (2)_{\omega}^{\text{FS}}$.*
- (2) *If $\dim(V) = \omega_2$, then $V \rightarrow (2)_{\omega_1}^{\text{FS}}$.*

Proof. Let $\mathcal{B} = \{b_i \mid i < \kappa\}$ (where κ is either ω_1 or ω_2 , accordingly) be a basis of V and $c : V \rightarrow \lambda$ (where λ is either ω or ω_1 , accordingly). We define $d : [\kappa]^2 \rightarrow \lambda$ by $d(\{\alpha, \beta\}) = c(b_\beta - b_\alpha)$ whenever $\alpha < \beta$. Then, since κ is measurable under AD and therefore $\kappa \rightarrow (\kappa)_{\lambda}^2$, there exists an $M \subseteq \kappa$ with $|M| = \kappa$ such that $[M]^2$ is d -monochromatic, say in colour i . Let $\alpha, \beta, \gamma \in M$; assume without loss of generality that $\alpha < \beta < \gamma$, and define $v = b_\beta - b_\alpha$ and $w = b_\gamma - b_\beta$. Then, we have

$$\begin{aligned} c(v) &= c(b_\beta - b_\alpha) = d(\{\alpha, \beta\}) = i \\ c(w) &= c(b_\gamma - b_\beta) = d(\{\beta, \gamma\}) = i \\ c(v + w) &= c(b_\gamma - b_\alpha) = d(\{\alpha, \gamma\}) = i, \end{aligned}$$

that is, $\text{FS}(\{v, w\})$ is c -monochromatic (in colour i). \square

Remark 2.17. Recall that, by Martin's theorem, $\omega_1 \rightarrow (\omega_1)_{2\omega}^2$. Therefore, by using the same argument as in the previous proof, we may conclude (under AD) that, for V a \mathbb{Q} -vector space of dimension ω_1 , we have $V \rightarrow (2)_{2\omega}^{\text{FS}}$.

3. OWINGS-LIKE RESULTS

We now turn to pairwise-sum configurations, in the spirit of Owings' problem. In contrast with the finite-sums phenomena studied in the previous section, the behaviour of $M + M$ under colourings exhibits a different interaction between descriptive-set-theoretic regularity and algebraic structure.

We begin by introducing the relevant notation and recalling the known background in ZFC. We then establish a positive result for \mathbb{R} under dependent choice, derived from regularity properties. Finally, we analyze algebraic limitations in cancellative groups and present positive results for large-dimensional \mathbb{Q} -vector spaces obtained from strong partition properties.

3.1. Pairwise-sum notation and background.

Definition 3.1. Let $(G, +)$ be a commutative semigroup, and let κ, θ be cardinals. We write $G \rightarrow (\kappa)_{\theta}^{+}$ to denote the statement that for every colouring of G in θ colours, there exists $M \subseteq G$, $|M| = \kappa$, such that $M + M$ is monochromatic.

In the previous definition, $M + M$ denotes the set $\{a + b \mid a, b \in M\}$. Owings's problem [18] asks whether $\mathbb{N} \rightarrow (\omega)_2^{+}$; this problem has been open since 1974 and, surprisingly, the main difficulty seems to lie on the requirement of using precisely two colours, since Hindman [8] has proved that $\mathbb{N} \rightarrow (\omega)_3^{+}$. Recently, there has been an important amount of work devoted to studying these types of Owings-type problems in the setting of the additive group \mathbb{R} , and nowadays it is known that $\mathbb{R} \rightarrow (\omega)_2^{+}$ holds [20], and that the statement $(\forall k < \omega)(\mathbb{R} \rightarrow (\omega)_k^{+})$ is both independent from [9], and consistent with [14, 20], the ZFC axioms.

So we proceed to study these kinds of statements, under ZF plus possibly other assumptions, for some particular cases of G (mostly either \mathbb{R} , or \mathbb{Q} -vector spaces with basis), for some specific configurations of κ, θ .

It is worth noting that results about the finite-finite configuration are already known completely, since [5, Theorem 1] completely characterizes those G for which it is the case that $(\forall n, r < \omega)(G \rightarrow (n)_r^{+})$. The proof of this result, belonging completely to the finite realm, does not use any version of the Axiom of Choice. In what follows, we explore other configurations.

3.2. Positive results on \mathbb{R} from regularity. Hindman, Leader and Strauss [9] proved, in the ZFC context, that if $N \subseteq \mathbb{R}$ is a Baire-measurable nonmeagre set, then there is an $X \in [\mathbb{R}]^{\omega_1}$ such that $X + X$ is monochromatic. Their proof only uses the Axiom of Choice when recursively choosing the “next” element of X , but the argument that such next element exists can be carried out in ZF only. Therefore we may assume a weaker hypothesis, such as DC, and use an entirely similar argument to obtain a countable such set X . We only sketch the proof for completeness, and refer the reader interested in some more specific details to [9, Lemma 4.1 and Theorem 4.2].

Lemma 3.2 (DC). *If $N \subseteq \mathbb{R}$ is nonmeagre and Baire-measurable, then there exists an $X \in [\mathbb{R}]^{\omega}$ such that $X + X \subseteq N$.*

Proof. Since N is Baire-measurable, we may write $N = A \triangle M$, with A open and M meagre. Without loss of generality, by replacing N with a translate if necessary, we may assume that $0 \in A$ (since, if we are able to find $X + X \subseteq c + N$, letting $Y = -\frac{c}{2} + X$ it is readily checked that $Y + Y \subseteq N$).

Pick a $\delta > 0$ such that $(0, 2\delta) \subseteq A$ and choose some $x_0 \in (0, \delta) \setminus \frac{1}{2}M$, it immediately follows that $\{x_0\} + \{x_0\} = \{2x_0\} \subseteq (0, 2\delta) \setminus M \subseteq N$. Using DC, we recursively choose the remaining sequence of x_n as follows: assuming by induction hypothesis that $X_n = \{x_k \mid k < n\}$ is such that $X_n + X_n \subseteq N$, we have

$$(0, \delta) \subseteq \frac{1}{2}A \cap \left(\bigcap_{k < n} (-x_k + A) \right),$$

which in turn ensures that $(0, \delta) \setminus L$ is meagre, where

$$L = \frac{1}{2}N \cap \left(\bigcap_{k < n} (-x_{k+N}) \right).$$

So it suffices to choose any $x_n \in ((0, \delta) \cap L) \setminus X_n$, and then the induction hypothesis will be preserved for $n + 1$, which allows the construction to continue. So in the end, the set $X = \{x_n \mid n < \omega\}$ is as sought. \square

The previous result yields the following positive Owings-like theorem for \mathbb{R} .

Theorem 3.3 (AD + DC). $\mathbb{R} \rightarrow (\omega)_{\omega}^{+}$.

Proof. Let $c : \mathbb{R} \rightarrow \omega$. Since $\mathbb{R} = \bigsqcup_{i < \omega} c^{-1}[\{i\}]$ is nonmeagre, there is an $i_0 < \omega$ such that $c^{-1}[\{i_0\}]$ is nonmeagre. Then Lemma 3.2 ensures the existence of an $X \in [\mathbb{R}]^{\omega}$ such that $X + X \subseteq c^{-1}[\{i_0\}]$. \square

This is a strong result, since it provides a countably infinite monochromatic set.

Remark 3.4. Notice that, assuming only AD, we can still prove that $\mathbb{R} \rightarrow (n)_{\omega}^{+}$ for every $n < \omega$. This is because, in Lemma 3.2, if one only wants to obtain a finite set X , one can do so in ZF without using DC.

3.3. Algebraic limitations and large-dimensional positives. We now contrast the descriptive-set-theoretic positive result above with algebraic restrictions in cancellative groups. In the ZFC context, the work in [5, Theorem 6] together with [17] imply that, if $(G, +)$ is an infinite abelian group without elements of order 4, then $G \not\rightarrow (2)_\omega^{+}$. We are able to prove the same result below, in ZF only, at the price of dropping some generality and obtaining a theorem only for \mathbb{Q} -vector spaces with basis.

Theorem 3.5. *Let V be a \mathbb{Q} -vector with basis. Then $V \not\rightarrow (2)_\omega^{+}$.*

Proof. Let \mathcal{B} be a basis of V , so $V \cong \bigoplus_{b \in \mathcal{B}} \mathbb{Q}$. As before, we define $\text{supp} : V \rightarrow [\mathcal{B}]^{<\omega}$ by $\text{supp}(v) = \{b \in \mathcal{B} \mid q_b \neq 0\}$ whenever $v = \sum_{b \in \mathcal{B}} q_b b$; so that $\text{supp}(v)$ is always a finite subset of \mathcal{B} . Now we let $c : V \rightarrow \mathbb{Q}$ be given by $c(v) = \sum_{b \in \mathcal{B}} q_b^2$, again, whenever $v = \sum_{b \in \mathcal{B}} q_b b$. Given any $v_0, v_1 \in V$, we prove that $X + X$, where $X = \{v_0, v_1\}$, is not monochromatic as follows: inject $\text{span}(v_0, v_1)$ into \mathbb{R}^n (where $n = |\text{supp}(v_1) \cup \text{supp}(v_0)|$), in the obvious way. So it suffices to prove that there do not exist two distinct vectors $a, b \in \mathbb{R}^n$ such that the elements of the set $\{2a, 2b, a + b\}$ all have the same Euclidean norm. If there were two such vectors a, b , we would have

$$4\|a\|^2 = \|a + b\|^2 = 4\|b\|^2.$$

Since a, b are distinct, they both have to be nonzero, so $\|a\| = \|b\| \neq 0$. Therefore $\|a\| + \|b\| = 2\|a\| = \|a + b\|$. Squaring both sides of the previous equation we get $\|a\|^2 + \|b\|^2 + 2\|a\|\|b\| = \|a\|^2 + \|b\|^2 + 2\langle a, b \rangle$, which implies that $\|a\|\|b\| = \langle a, b \rangle$. The Cauchy–Schwarz inequality then tells us that the equation holds if and only if $b = \lambda a$ for some $\lambda \in \mathbb{R}$, however this implies that $\|a\| = \|b\| = \|\lambda a\| = |\lambda|\|a\|$, so $\lambda = \pm 1$. This is a contradiction, as it implies either $a = b$, or $a + b = 0$. \square

On the other hand, Leader and Russell showed that if V is a \mathbb{Q} -vector space of sufficiently large dimension (at least \beth_ω), then $V \rightarrow (\omega)_\theta^{+}$ for every natural number θ [16, Theorem 2]. Utilizing several very similar ideas, we are able to utilize the Axiom of Determinacy to derive the following positive result from the strong combinatorial properties of ω_1 and ω_2 . We omit some details regarding the behaviour of the π_i , since these work exactly as in the proof of [16, Theorem 2].

Theorem 3.6 (AD). *Let V be a \mathbb{Q} -vector space of dimension $\kappa \in \{\omega_1, \omega_2\}$. Then*

$$\forall \theta \in \mathbb{N} \quad (V \rightarrow (\omega)_\theta^{+}).$$

Proof. Let $\mathcal{B} = \{b_j \mid j < \kappa\}$ be a basis for V and let $c : V \rightarrow \theta$ be arbitrary. We begin by defining the following sequences for $i < \theta + 1$,

$$\pi_i := (\overbrace{2, \dots, 2}^{i \text{ times}}, \overbrace{1, \dots, 1}^{2(\theta-i) \text{ times}}) \in \{1, 2\}^{2\theta-i}$$

Notice that $L := \max_{i < \theta+1} |\pi_i| = \max_{i < \theta+1} (2\theta - i) = 2\theta$, so

$$\pi'_i := (\overbrace{0, \dots, 0}^{L-|\pi_i|}) \frown \pi_i \in 3^L,$$

and we also define

$$P'_i := \left\{ \sum_{k < L} (\pi'_i)_k b_{j_k} \mid j_0 < \dots < j_{L-1} < \kappa \right\}.$$

We define $\theta + 1$ colourings $c_i : [\kappa]^L \rightarrow \theta$ by

$$c_i(\{j_0 < \dots < j_{L-1}\}) = c \left(\sum_{k < L} (\pi'_i)_k b_{j_k} \right).$$

Now consider the colouring $d : [\kappa]^L \rightarrow \theta^{\theta+1}$ given by $d(S) = (c_i(S))_{i < \theta+1}$. By the Ramsey-type theorem on measurable cardinals, $\kappa \rightarrow (\kappa)_{\theta^{\theta+1}}^L$, we can find an $N \subseteq \kappa$ of cardinality κ and fixed $I_0, \dots, I_\theta < \theta$ such that $[N]^L$ is monochromatic in colour (I_0, \dots, I_θ) . By the pigeonhole principle, there are $i_1 < i_2 < \theta + 1$ such that $I_{i_1} = I_{i_2} =: I$. Define $M := \max_{j < 2}(L - |\pi_{i_j}|)$, enumerate $\mathcal{B}' := \{b_j \mid j \in N\} = \{b'_j \mid j < \kappa\}$, and let

$$x_n := \sum_{r < M} 0b'_r + \sum_{r < i_1} b'_{M+r} + \sum_{r < i_2 - i_1} b'_{\omega(r+1)+n} + \sum_{r < 2(\theta - i_2)} \frac{1}{2} b'_{\omega(i_2 - i_1 + 1) + r}.$$

for $n < \omega$. Finally, let $X := \{x_n \mid n < \omega\}$. Then, for $m, n < \omega$, one can observe that $x_m + x_n \in P_{i_1} \cup P_{i_2}$, depending on whether $m \neq n$ or $m = n$; in any case, this implies that $c(x_m + x_n) = I$. Therefore, $X + X$ is monochromatic, as intended. \square

4. CONCLUSIONS

We will make explicit which fragments of set-theoretic strength are used in the different parts of the paper. The results above arise from three logically distinct sources: Dependent Choice, determinacy-driven regularity on the reals and partition properties at ω_1 and ω_2 , and purely algebraic rigidity.

Results in ZF: The finite-finite configurations, both for Hindman and Owings-type theorems (Theorem 2.3 and [5, Theorem 1], respectively) can be carried out without any assumptions additional to ZF. The infinite-finite configuration for Hindman-type theorems (Theorem 2.4) also works in ZF for Dedekind-infinite Abelian groups, or in ZF plus the assumption that every infinite set is Dedekind-infinite if one wants the theorem to be stated for all infinite Abelian groups.

Most notably, the negative Hindman-like results in the real line (regarding the uncountable-finite and finite-infinite configurations), i.e. Theorem 2.8 and Corollary 2.9, also hold in ZF only. Finally, for both Hindman-like and Owings-like problems, we obtain negative results in the finite-infinite configurations (Theorems 2.11 and 3.5), for \mathbb{Q} -vector spaces with basis, and these proofs can be carried out in ZF.

Results under DC: Our version of the Δ -system lemma, Lemma 2.7, uses only the Axiom of Dependent Choice, and it provides the combinatorial control of supports that is used in the proof of the negative Hindman-like result for the uncountable-finite configuration on \mathbb{Q} -vector spaces (Theorem 2.10).

Likewise, Lemma 3.2 (Hindman–Leader–Strauss) relies only on Dependent Choice; although in order to apply this lemma to obtain a positive Owings-type theorem will require more assumptions.

Results under AD: It seems that AD provides us with many positive Ramsey-theoretic results, in contexts in which such results are negative under the usual ZFC axioms. Specifically regarding this paper, the positive results in \mathbb{Q} -vector spaces of dimension ω_1 or ω_2 depend essentially on consequences

of $\text{ZF} + \text{AD}$. Theorems 2.14 and 2.15 assert that, under determinacy, ω_1 and ω_2 satisfy strong partition relations analogous to those of measurable cardinals. These partition properties are used directly in the proofs of the positive Hindman and Owings-type results that we obtain (for the 2-infinite Hindman configuration, Theorem 2.16; and for the infinite-finite Owings configuration, Theorem 3.6).

Regularity of the reals under $\text{AD} + \text{DC}$: Assuming $\text{AD} + \text{DC}$ provides us with a very powerful, positive, Owings-type result in the infinite-infinite configuration for \mathbb{R} (Theorem 3.3). This is, once again, in sharp contrast with the known ZFC results, where such statements are known to be independent of the axioms.

This work provides a systematic comparison between finite-sums and pairwise-sums partition relations in the setting of ZF , plus additional assumptions such as DC or AD (or both). For finite-sums configurations, the uncountable phenomenon fails in cancellative structures such as \mathbb{Q} -vector spaces of uncountable dimension, confirming that the classical Hindman behaviour is essentially countable in this algebraic context. At the same time, determinacy restores strong partition properties at ω_1 and ω_2 , yielding positive finite configurations in sufficiently large dimensions.

For pairwise-sums configurations, the situation differs substantially. Regularity properties available under determinacy ensure that every countable colouring of \mathbb{R} admits an infinite monochromatic sumset. However, algebraic constraints once again impose strict limitations in highly cancellative groups, while large-dimensional vector spaces regain positive behaviour through determinacy-driven partition principles.

The picture that emerges is that infinite Ramsey phenomena in the absence of Choice are governed by a balance between algebraic structure and set-theoretic strength. Determinacy supplies large-cardinal-like partition properties at specific cardinals, but algebraic rigidity can still prevent uncountable homogeneity. Understanding precisely which algebraic conditions permit positive partition relations under weak choice principles remains a natural direction for further investigation.

In particular, it would be of interest to determine whether stronger uncountable versions of Owings-type configurations can be obtained under $\text{ZF} + \text{AD}$, and to clarify the extent to which the positive results presented here depend essentially on determinacy rather than on weaker regularity assumptions.

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