THE ADJACENT HINDMAN'S THEOREM AND THE $\mathbb{Z}$-RAMSEY THEOREM

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Abstract. We consider the weakening of Ramsey's theorem that arises from considering only translation-invariant colourings of pairs, and show that this has the same strength (both from the viewpoint of Reverse Mathematics and from the viewpoint of Computability Theory) as the *adjacent Hindman's theorem*, proposed by L. Carlucci (Arch. Math. Log. **57** (2018), 381–359). We also investigate some higher dimensional versions of both of these statements.

1. Introduction

In this paper, we consider some Ramsey-theoretic combinatorial results from the perspective of both Computability Theory and Reverse Mathematics. Ramsey's theorem on dimension n, denoted RT^n ($n \geq 1$), is the statement that for every $c : [\mathbb{N}]^n \longrightarrow k$ (where k is an arbitrary finite number), there exists an infinite set $X \subseteq \mathbb{N}$ such that $[X]^n$ is c-monochromatic. On the other hand, Hindman's theorem, denoted HT, is the statement that for every $c : \mathbb{N} \longrightarrow k$ there exists an infinite set $X \subseteq \mathbb{N}$ such that the set

$$\mathrm{FS}(X) = \left\{ \sum_{x \in F} x \,\middle|\, F \subseteq X \text{ is finite and nonempty} \right\}$$

is c-monochromatic.

These two principles have been extensively studied, and constitute an important vein of contemporary research in the interface between combinatorics and (various branches of) logic. For example, it is known that, if $n \geq 3$, then RT^n is equivalent to ACA_0 over RCA_0; somewhat surprisingly, RT^2 is a principle at the same time strictly weaker than ACA_0 and strictly stronger than WKL_0, whereas RT^1 is simply the pigeonhole principle (strictly stronger than plain RCA_0 but much weaker than WKL_0). On the other hand, it is also known that, again over RCA_0, ACA_0^+ implies HT which in turn implies ACA_0, where ACA_0^+ is essentially ACA_0 plus the existence of the ω-th Turing jump of every set. The precise strength of HT in the hierarchy of reverse-mathematical principles is still unknown, and determining this strength is currently one of the important open problems in Reverse Mathematics.

With the finer distinctions provided by Computability Theory, especially the theory of Turing reducibility, one can be more precise about the results just mentioned. For example, for Ramsey's theorem, it is known [8] that for every *instance* c of the problem (that is, every colouring) there exists a solution (a set that is monochromatic for the colouring in question) X such that $X \leq_T c^{(2n+2)}$, the $(2n+2)$-fold Turing jump of c; on the other hand, there is a computable instance c of Ramsey's theorem such that every solution X satisfies $\varnothing^{(n-2)} \leq_T X$. In the case of Hindman's theorem, it is known [1] that there are computable instances c to the theorem such that every monochromatic solution X satisfies $\varnothing' \leq_T X$ whereas every instance c of the theorem admits a solution X such that $X \leq_T c^{(\omega+1)}$.

In this paper, we consider certain weakenings of each of the two principles mentioned above. On the front of Hindman's theorem, we shall work with L. Carlucci's *Adjacent Hindman's Theorem* proposed in [2], which results from weakening the monochromaticity requirement not to a full set $\mathrm{FS}(X)$ but rather to the restricted set $\mathrm{AFS}(X)$ of *adjacent* finite sums, meaning those sums of finitely many *consecutive* elements of X when ordered increasingly, leaving no gaps. Carlucci showed that, over RCA_0, RT^2 implies AHT, even with an extra condition that we will also consider, called *apartness*. On the other hand, we will consider a weakening of Ramsey's theorem that arises from restricting the allowable problems for which we require a solution —concretely, stating the theorem only for those colourings that are invariant under the translation action of $\mathbb{Z}$, obtaining what we will call the $\mathbb{Z}$-*Ramsey's theorem*. Although this principle has not been previously studied within Reverse Mathematics, the idea of restricting instances of Ramsey's theorem only to translation-invariant colourings has already been used, e.g., by Petrenko and Protasov in their definition of $\mathbb{Z}$-Ramsey ultrafilters [9].

Many of the principles mentioned in the previous paragraph turn out to be equivalent to ACA_0 over RCA_0, so for the most part the viewpoint of Reverse Mathematics will not be very informative in that respect. In view of this, we focus more on the Computability Theoretic aspect of the principles, thinking instead about the *reducibility* relation by means of which an instance of one problem can be turned into an instance of a different problem and a solution to the latter gets turned back into a solution to the original instance. More concretely, we will work with *uniform reducibility*, sometimes also known as (or a particular case of) Weihrauch reducibility[1]. Recall [7] that a problem

[1]In fact, the proofs of the theorems in this paper actually establish what is known as a *strong Weihrauch reducibility*, since e.g. the Turing functionals for turning a solution to one problem into a solution of the other do not require

P is **uniformly reducible** to another problem Q, denoted $P \leq_u Q$, if there are Turing functionals Φ, Ψ such that for every instance X of P, Φ^X is an instance of Q, and for every solution Y to Φ^X, $\Psi^{X \oplus Y}$ is a solution to X. So we will use this notion of reducibility to gauge the position of these principles within the computability-theoretic hierarchy; all of our proofs can be turned, by simply looking at them carefully enough, into proofs in RCA_0, thus yielding corresponding (but coarser) Reverse Mathematics results as corollaries.

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2. The $\mathbb{Z}$-invariant Ramsey theorem

Definition 1.

(1) Given an $n \in \mathbb{N}$, we will say that a colouring of n-tuples $c : [\mathbb{N}]^n \longrightarrow k$ is **$\mathbb{Z}$-invariant** if, for every n-tuple $\{x_1, \dots, x_n\} \in [\mathbb{N}]^n$ and for every $z \in \mathbb{N}$, it is the case that $c(\{x_1, \dots, x_n\}) = c(\{x_1 + z, \dots, x_n + z\})$.

(2) The **$\mathbb{Z}$-Ramsey theorem for n-tuples**, which we will denote by $\mathbb{Z}\text{-}\mathsf{RT}^n$, is the statement that every $\mathbb{Z}$-invariant colouring of n-tuples admits an infinite monochromatic set.

It is clear that, for each $n \in \mathbb{N}$, $\mathbb{Z}\text{-}\mathsf{RT}^n$ is a weaker statement than RT^n. The following theorem basically establishes that $\mathbb{Z}\text{-}\mathsf{RT}^{n+1}$ implies RT^n over RCA_0, although, as we warned in the introduction, we will attempt to provide more precise information by phrasing our theorem in the language of uniform reducibility.

Theorem 2. *For each $n \geq 2$, $\mathsf{RT}^n \leq_u \mathbb{Z}\text{-}\mathsf{RT}^{n+1}$.*

Proof. The witnessing Turing functionals, which we denote by Φ and Ψ, are as follows: for a colouring $c : [\mathbb{N}]^n \longrightarrow 2$ we let $\Phi^c : [\mathbb{N}]^{n+1} \longrightarrow 2$ be given by

$$\Phi^c(x_0, x_1, \dots, x_n) = c(x_1 - x_0, x_2 - x_0, \dots, x_n - x_0).$$

information from the concrete instance of the problem at hand (more concretely, in the notation explained below we have that Ψ^X, rather than $\Psi^{X \oplus Y}$, is already a solution to Y). We are thankful to L. Carlucci for this observation.

It is easily verified that Φ^c is $\mathbb{Z}$-invariant. In the other direction, given an infinite set $A \subseteq \mathbb{N}$, we let

$$\Psi^A = \{x - x_0 \big| x \in A \setminus \{x_0\}\},$$

where $x_0 = \min(A)$. If $[A]^{n+1}$ is Φ^c-monochromatic (say, in colour i) then $[\Psi^A]^n$ is c-monochromatic (in the same colour i), since for any elements $x_1 - x_0, \dots, x_n - x_0 \in \Psi^A$ we have

$$c(x_1 - x_0, \dots, x_n - x_0) = \Phi^c(x_0, x_1, \dots, x_n) = i.$$

□

Since the statements RT^n are equivalent for $n \geq 3$ (and they are equivalent to ACA_0), the only interesting corollary, from the perspective of Reverse Mathematics, is the part where $n \leq 3$.

Corollary 3. *Over* RCA_0 *we have:*

(1) $\mathbb{Z}$-RT^n *is equivalent to* ACA_0 *for each* $n \geq 4$*, and*

(2) $\mathsf{ACA}_0 \Rightarrow \mathbb{Z}\text{-}\mathsf{RT}^3 \Rightarrow \mathsf{RT}^2 \Rightarrow \mathbb{Z}\text{-}\mathsf{RT}^2 \Rightarrow \mathsf{RT}^1$.

Later on, in the last section (and once we have more tools under our belt), we come back to whether the first two arrows from part (2) of Corollary 3 are reversible.

2.1. **Relation with the Adjacent Hindman's Theorem.** Let us begin by recalling Carlucci's adjacent Hindman's theorem.

Definition 4. The **adjacent Hindman's theorem**, denoted by AHT, is the statement that, for every finite colouring $c : \mathbb{N} \longrightarrow k$, there exists an infinite set X such that the set

$$\mathrm{AFS}(X) = \{x_n + x_{n+1} + \cdots + x_{n+l} \big| n, l \in \mathbb{N}\}$$

is monochromatic, where the sequence $(x_n \big| n \in \mathbb{N})$ represents the (unique) increasing enumeration of the set X.

Given that the set $\mathrm{AFS}(X)$ is defined in such a way that the order within our set matters (unlike the usual FS sets utilized in Hindman's theorem), one could conceivably state a version of the AHT for *sequences* rather than sets, i.e. affirming the existence of a sequence $(x_n \big| n \in \mathbb{N})$, not necessarily increasing (or even injective!) whose adjacent finite sums are monochromatic. However, it is readily seen that these two versions of the AHT must be equivalent (whether from the viewpoint of Reverse Mathematics over RCA_0, or from the viewpoint of Computability Theory). Clearly the original version as stated by Carlucci implies the "sequence" version (by replacing the set X with the sequence that increasingly enumerates its elements); conversely, given an arbitrary sequence x_n, one can recursively define $y_1 = x_1$, $k_1 = 1$; and $y_{n+1} = x_{k_n+1} + x_{k_n+2} + \cdots + x_{k_{n+1}}$, with k_{n+1} the least number making the y_{n+1} thus defined to be strictly larger than y_n. This way we obtain a strictly increasing sequence $(y_n \big| n \in \mathbb{N})$ (in particular, we obtain a set Y whose increasing enumeration is precisely the sequence of y_n) such that $\mathrm{AFS}(Y) = \mathrm{AFS}(y_n \big| n \in \mathbb{N}) \subseteq \mathrm{AFS}(x_n \big| n \in \mathbb{N})$.

The following theorem is the observation that originally launched the work in this paper.

Theorem 5. *$\mathbb{Z}$-$\mathsf{RT}^2 \leq_u \mathsf{AHT}$ and $\mathsf{AHT} \leq_u \mathbb{Z}$-$\mathsf{RT}^2$.*

Proof. For the first reducibility relation, we describe the corresponding Turing functionals Φ_1, Ψ_1 as follows. Given a $\mathbb{Z}$-invariant colouring $c : [\mathbb{N}]^2 \longrightarrow k$, $\Phi_1^c : \mathbb{N} \longrightarrow k$ is defined by $\Phi_1^c(y) = c(0, y)$. Note that the assumption that c is $\mathbb{Z}$-invariant implies $\Phi_1^c(y) = c(x, x+y)$ for every natural number x. On the other hand, for any infinite set $Y = \{y_n \big| n \in \mathbb{N}\}$ (where the indexing y_n of the elements of Y constitutes an increasing enumeration), we let $\Psi_1^Y = \{y_1 + \cdots + y_n \big| n \in \mathbb{N}\}$. For every pair of elements of Ψ_1^Y, we have

$$\begin{aligned} c(y_1 + \cdots + y_n, y_1 + \cdots + y_m) &= c(y_1 + \cdots + y_n, (y_1 + \cdots + y_n) + (y_{n+1} + \cdots + y_m)) \\ &= c(0, y_{n+1} + \cdots + y_m) \\ &= \Phi_1^c(y_{n+1} + \cdots + y_m). \end{aligned}$$

Since numbers of the form $y_{n+1} + \cdots + y_m$ are precisely the elements of $\mathrm{AFS}(Y)$, the conclusion is that $\mathrm{AFS}(Y)$ is Φ_1^c-monochromatic if and only if $[\Psi_1^Y]^2$ is c-monochromatic.

For the converse reducibility relation, we denote the relevant functionals by Φ_2, Ψ_2. Given a $d : \mathbb{N} \longrightarrow k$ we let $\Phi_2^d : [\mathbb{N}]^2 \longrightarrow k$ be given by $\Phi_2^d(\{x, y\}) = d(y - x)$, whenever $x < y$; it is readily seen that Φ_2^d is a $\mathbb{Z}$-invariant colouring. On the other hand, for an infinite set X, first define recursively $X' = \{x_n \big| n \in \mathbb{N}\}$ by letting x_0, x_1 be the two smallest elements of X, and then we let each x_{n+1} be the least element of X so that $x_{n+1} - x_n > x_n - x_{n-1}$. We then let

$\Psi_2^X = \{x_{n+1} - x_n \big| n \in \mathbb{N}\}$. Note that an element of $\mathrm{AFS}(\Psi_2^X)$ is of the form

$$(x_{n+1} - x_n) + (x_{n+2} - x_{n+1}) + \cdots + (x_{n+k+1} - x_{n+k}) = x_{n+k+1} - x_n,$$

therefore if $x_{n+k+1} - x_n = y \in \mathrm{AFS}(\Psi_2^X)$ then $d(y) = d(x_{n+k+1} - x_n) = \Phi_2^d(x_n, x_{n+k+1})$; so, if X is Φ_2^d-monochromatic then $\mathrm{AFS}(\Psi_2^X)$ is d-monochromatic. □

Note that the previous proof, in a sense, shows that $\mathbb{Z}$-invariant colourings are precisely those colourings that depend only on the distance between the two elements of the pair being coloured. Of course, we now have the following corollary, whose proof consists of carefully carrying out the previous proof within Recursive Arithmetic.

Corollary 6. *Over* RCA_0*,* AFS *is equivalent to* $\mathbb{Z}$-RT^2*.*

3. Adjacent Hindman's theorem in higher dimensions.

We now proceed to study a higher-dimensional version of the Adjacent Hindman's Theorem, much in the same spirit that the Milliken–Taylor theorem constitutes a two-dimensional version of the usual Hindman's theorem. Recall that the Milliken–Taylor theorem ensures that, given any colouring of pairs of finite subsets of $\mathbb{N}$, there exists a pairwise disjoint family such that all *ordered* pairs of finite unions have the same colour. In the adjacent context, we will have to consider *adjacent* pairs of adjacent finite unions, and similarly for higher dimensions.

Definition 7.

(1) Let $\vec{x} = \langle x_n \big| n \in \mathbb{N}\rangle$ be a sequence of natural numbers, and let $d \in \mathbb{N} \setminus \{0\}$. We define the set of **adjacent d-tuples of adjacent sums from $\vec{x}$** to be the set

$$\mathrm{AFS}^d(\vec{x}) = \left\{ \left(\sum_{k=k_0}^{k_1} x_k, \sum_{k=k_1+1}^{k_2} x_k, \dots, \sum_{k=k_{d-1}+1}^{k_d} x_k \right) \Bigg| k_0 \le k_1 < k_2 < \cdots < k_d \right\}$$

(2) We define the **d-Adjacent Hindman's Theorem**, denoted AHT^d, to be the statement that for every colouring $d : \mathbb{N}^d \longrightarrow k$ there exists an infinite set $Y \subseteq \mathbb{N}$ such that, if $\vec{y} = \langle y_n \big| n \in \mathbb{N}\rangle$ is the increasing enumeration of Y, then the set $\mathrm{AFS}^d(\vec{y})$ is d-monochromatic.

Just as in the case of the 1-dimensional adjacent Hindman's theorem, AHT^d for $d > 1$ could be phrased directly in terms of sequences, and we would still obtain an equivalent statement. The following is the generalization of Theorem 5 to this context.

Theorem 8. *For each $d \in \mathbb{N} \setminus \{0\}$, we have $\mathbb{Z}\text{-}\mathsf{RT}^{d+1} \leq_u \mathsf{AHT}^d$ and $\mathsf{AHT}^d \leq_u \mathbb{Z}\text{-}\mathsf{RT}^{d+1}$.*

Proof. The Turing functionals Φ_1, Ψ_1 witnessing that $\mathbb{Z}\text{-}\mathsf{RT}^{d+1} \leq_u \mathsf{AHT}^d$ are: for a $\mathbb{Z}$-invariant colouring $c : [\mathbb{N}]^{d+1} \longrightarrow k$, we let $\Phi_1^c : \mathbb{N}^d \longrightarrow k$ be given by

$$\Phi_1^c(y_1, \dots, y_d) = c(0, y_1, y_1 + y_2, \dots, y_1 + y_2 + \dots + y_d).$$

On the other hand, given an infinite set $Y = \{y_n \big| n \in \mathbb{N}\}$, enumerated increasingly, let $\Psi_1^Y = \{y_1 + \dots + y_n \big| n \in \mathbb{N}\}$. Now, if $x_0, \dots, x_d \in \Psi_1^Y$ are distinct elements, with $x_0 < \dots < x_d$, then we have $x_i = y_1 + \dots + y_{k_i}$ for each i, with $k_0 < k_2 < \dots < k_d$. Then, by the $\mathbb{Z}$-invariance of c we have

$$\begin{aligned} c(x_0, \dots, x_d) &= c(0, x_1 - x_0, x_2 - x_0, \dots, x_d - x_0) \\ &= c(0, z_1, z_1 + z_2, \dots, z_1 + z_2 + \dots + z_d) \\ &= \Phi_1^c(z_1, z_2, \dots, z_d), \end{aligned}$$

where we have defined $z_i = x_i - x_{i-1} = y_{k_{i-1}+1} + \dots + y_{k_i}$. So the values of c on $(d+1)$-tuples from Ψ_1^Y are exactly the values of Φ_1^c on d-tuples from Y that have the form

$$(y_{k_0+1} + \dots + y_{k_1}, y_{k_1+1} + \dots + y_{k_2}, \cdots, y_{k_{d-1}+1} + \dots + y_{k_d}),$$

which are precisely the elements of $\mathrm{AFS}^d(Y)$. Hence Ψ_1^Y is c-monochromatic if and only if Y is Φ_1^c-monochromatic.

Now, to prove $\mathsf{AHT}^d \leq_u \mathbb{Z}\text{-}\mathsf{RT}^{d+1}$, the witnessing Turing functionals will be Φ_2, Ψ_2 defined as follows: for $d : \mathbb{N}^d \longrightarrow k$ we let $\Phi_2^d : [\mathbb{N}]^{d+1} \longrightarrow k$ be given by $\Phi_2^d(x_0, \dots, x_d) = d(x_1 - x_0, x_2 - x_1, \dots, x_d - x_{d-1})$ (if $x_1 < \ldots < x_{d+1}$); it is easy to see that Φ_2^d is a $\mathbb{Z}$-invariant colouring. For the definition of Ψ_2 we proceed as in the proof of theorem 5: given an infinite set X, first define $X' = \{x_n \big| n \in \mathbb{N}\}$ such that x_0, x_1 are the two smallest elements of X, and x_{n+1} is the least element of X so that $x_{n+1} - x_n > x_n - x_{n-1}$ for each $n \geq 1$. We then let $\Psi_2^X = \{x_{n+1} - x_n \big| n \in \mathbb{N}\}$. Note that, if

$y_j = x_{j+1} - x_j$, then an element $\vec{y} \in \mathrm{AFS}^d(\Psi_2^X)$ is of the form

$$\begin{aligned}\vec{y} &= (y_{i_0} + \cdots + y_{i_1-1}, y_{i_1} + \cdots + y_{i_2-1}, \ldots, y_{i_{d-1}} + \cdots + y_{i_d-1}) \\ &= (x_{i_1} - x_{i_0}, x_{i_2} - x_{i_1}, \ldots, x_{i_d} - x_{i_{d-1}}),\end{aligned}$$

so that $d(\vec{y}) = \Phi_2^d(x_0, \ldots, x_d)$ and hence, if $[X]^{n+1}$ is Φ_2^d-monochromatic this implies that $\mathrm{AFS}(\Psi_2^X)$ is d-monochromatic. □

As clearly as always, this leads also to the following.

Corollary 9. *For each d, the statements $\mathbb{Z}$-RT^{d+1} and AHT^d are equivalent over $\mathsf{RCA_0}$.*

3.1. **The apartness condition.** In Carlucci's paper [2] (where the original formulation of the AHT can be found), a version of the AHT is considered where one requires an *apartness* condition on the generators of the monochromatic set. In order to understand this requirement, recall that given an $x \in \mathbb{N}$ one defines $\lambda(x)$ to be the unique n such that $2^n \mid x$ but $2^{n+1} \nmid x$; on the other hand, one defines $\mu(x) = \lfloor \log_2(x) \rfloor$. Pictorially, it is easier to remember that, when expressing x in binary notation, $\lambda(x)$ provides the position of the first non-zero digit and $\mu(x)$ is the position of the last (non-zero) digit (first and last meaning from lower to higher powers of 2 —i.e., when reading the number from right to left); even more intuitively, one can simply think of the natural number n as a finite subset of $\mathbb{N}$ (by letting the binary expansion of n represent the characteristic function of such finite subset) and then $\lambda(n), \mu(n)$ are simply the minimum and maximum elements of this finite set.

Definition 10. A sequence $\langle x_n \big| n \in \mathbb{N} \rangle$ is said to satisfy *the apartness condition* if, for every $n \in \mathbb{N}$, we have $\mu(x_n) < \lambda(x_{n+1})$.

The reader familiar with the definition of a *block sequence* on $[\mathbb{N}]^{<\aleph_0}$ will recognize that a sequence of natural numbers satisfies the apartness condition precisely when the finite sets corresponding to the elements of the sequence (as described in the previous paragraph) form a block sequence. Carlucci's *adjacent Hindman's theorem with apartness* is the principle stating the same as AHT but requiring that the sequence whose set of adjacent finite sums is monochromatic satisfy the apartness condition; we will henceforth denote this principle with the symbol AHT_a. We similarly define the higher-dimensional versions AHT_a^d for each d.

Recall the proof of Theorem 5. Looking at the definitions of the relevant Turing functionals Ψ_1, Ψ_2, the reader will notice that, if the infinite set $Y = \{y_n \big| n \in \mathbb{N}\}$ satisfies the apartness condition, then the set Ψ_1^Y will have an increasing enumeration $\{x_n \big| n < \mathbb{N}\}$ satisfying $\mu(x_n - x_{n-1}) < \lambda(x_{n+1} - x_n)$. Conversely, given an infinite set $X = \{x_n \big| n < \mathbb{N}\}$ (increasing enumeration) satisfying $\mu(x_n - x_{n-1}) < \lambda(x_{n+1} - x_n)$, then the set Ψ_2^X, as defined in the proof of Theorem 5, which is simply $\{x_{n+1} - x_n \big| n \in \mathbb{N}\}$ (note that, in this case, the auxiliary subset $X' \subseteq X$ is simply X itself), will satisfy the apartness condition. This motivates the following definition of a condition that is to versions of Ramsey's theorem what the apartness condition is to versions of Hindman's theorem.

Definition 11.

(1) A sequence $\langle x_n \big| n \in \mathbb{N}\rangle$ is said to satisfy the **separation condition** if, for every $n \in \mathbb{N}$, we have $\mu(x_n - x_{n-1}) < \lambda(x_{n+1} - x_n)$.

(2) Given an $n \in \mathbb{N}$, the **ℤ-Ramsey theorem for n-tuples with separation** is the statement that every ℤ-invariant colouring of n-tuples admits an infinite monochromatic set that, when enumerated increasingly, yields a sequence satisfying the separation condition. We denote this principle with the symbol $\mathbb{Z}\text{-}\mathsf{RT}_s^n$.

Clearly, $\mathbb{Z}\text{-}\mathsf{RT}_s^n$ implies $\mathbb{Z}\text{-}\mathsf{RT}^n$ in RCA_0 or, in terms of uniform reducibility, we have $\mathbb{Z}\text{-}\mathsf{RT}^n \leq \mathbb{Z}\text{-}\mathsf{RT}_s^n$ (intuitively, requiring the separation condition in the monochromatic set makes the corresponding statement stronger). Looking carefully (whether literally as Computability statements, or as proofs in RCA_0) at the proofs from Theorem 8 (as described two paragraphs above but now in this more general context), yields the following.

Theorem 12.

(1) $\mathbb{Z}\text{-}\mathsf{RT}_s^{d+1} \leq_u \mathsf{AHT}_a^d$ *and* $\mathsf{AHT}_a^d \leq_u \mathbb{Z}\text{-}\mathsf{RT}_s^{d+1}$,

(2) for each d, the statements $\mathbb{Z}\text{-}\mathsf{RT}_s^{d+1}$ *and* AHT_a^d *are equivalent over* RCA_0.

For a relationship between the versions of $\mathbb{Z}\text{-}\mathsf{RT}$ with or without the separation condition (or, equivalently, between the versions of AHT with or without apartness), Carlucci [2] proves that RT^2 implies AHT_a. Using essentially the same reasoning, we prove the higher-dimensional general version of this result.

Theorem 13. *For each $n \in \mathbb{N}$, we have $\mathsf{AHT}_a^n \leq_u \mathsf{RT}^{n+1}$.*

Proof. Consider the Turing functionals Φ and Ψ given as follows: for a colouring $c : [\mathbb{N}]^n \longrightarrow k$, let $\Phi^c : [\mathbb{N}]^{n+1} \longrightarrow k$ be given by

$$\Phi^c(x_0, \dots, x_n) = c(2^{x_1} + 2^{x_1+1} + \cdots + 2^{x_2-1}, 2^{x_2} + 2^{x_2+1} + \cdots + 2^{x_3-1}, \dots, 2^{x_{n-1}} + 2^{x_{n-1}+1} + \cdots + 2^{x_n}).$$

On the other hand, given a $Y = \{y_n \big| n \in \mathbb{N}\}$ enumerated increasingly, let

$$\Psi^Y = \{2^{y_n} + 2^{y_n+1} + \cdots + 2^{y_{n+1}} \big| n \in \mathbb{N}\}.$$

Then $[Y]^{n+1}$ is Φ^c-monochromatic if and only if $\mathrm{AFS}^n(\Psi^Y)$ is c-monochromatic. $\square$

This way, we obtain the following diagram, where the arrows can mean either implication under RCA_0, or uniform reducibility (of the item to the right of the arrow to the item to the left of it)

$$\begin{array}{ccccccccccccccc}
\cdots & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}_s^{n+1} & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}^{n+1} & \Longrightarrow & \mathsf{RT}^n & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}_s^n & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}^n & \Longrightarrow & \mathsf{RT}^{n-1} & \Longrightarrow & \cdots \\
 & & \Updownarrow & & \Updownarrow & & & & \Updownarrow & & \Updownarrow & & & & \\
\cdots & & \mathsf{AHT}_a^n & \Longrightarrow & \mathsf{AHT}^n & & & & \mathsf{AHT}_a^{n-1} & \Longrightarrow & \mathsf{AHT}^{n-1} & & & & \cdots
\end{array}$$

Towards the right-extreme of this diagram (which extends infinitely to the left), we find the following configuration:

$$\begin{array}{ccccccccccccccc}
\cdots & \Longrightarrow & \mathsf{RT}^3 & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}_s^3 & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}^3 & \Longrightarrow & \mathsf{RT}^2 & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}_s^2 & \Longrightarrow & \mathbb{Z}\text{-}\mathsf{RT}^2 & \Longrightarrow & \mathsf{RT}^1 \\
 & & & & \Updownarrow & & \Updownarrow & & & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\cdots & & & & \mathsf{AHT}_a^2 & \Longrightarrow & \mathsf{AHT}^2 & & & & \mathsf{AHT}_a & \Longrightarrow & \mathsf{AHT} & & \text{pigeonhole}
\end{array}$$

This is very informative from the perspective of uniform reducibility. From the perspective of Reverse Mathematics, on the other hand, a huge portion of the diagram collapses, since it is known that RT^3 already implies ACA_0. Therefore, the first diagram consists of equivalent statements, and the only question remaining should be regarding the second diagram. The next (and last) section of the paper addresses this question.

4. A lower bound and questions.

In this section, we show that AHT^2 implies ACA_0 over RCA_0. Thus, the leftmost five nodes of the last shown diagram collapse, all of them being equivalent, and really the only point of separation happens in the arrow connecting $\mathbb{Z}$-RT^3 to RT^2. The following result, stated in both the language of Computability Theory and of Reverse Mathematics, is proved with much the same ideas as the classical proof that RT^3 implies ACA_0.

Theorem 14.

(1) There exists a computable instance of AHT^2 *whose solutions compute* $0'$.

(2) Over RCA_0, AHT^2 *(equivalently,* $\mathbb{Z}$-RT^3*) implies* ACA_0.

Proof. Consider the colouring $c : \mathbb{N}^2 \longrightarrow 2 \times 2$ given by letting $c(x, y) = (i, j)$, where i equals 1 if and only if $\lambda(x) < \lambda(y)$, and j equals 1 if and only if $(0' \cap \lambda(x))[\mu(x)] = (0' \cap \lambda(x))[\mu(y)]$. Suppose that $\vec{x} = (x_n \big| n \in \mathbb{N})$ is a sequence such that $\mathrm{AFS}^2(\vec{x})$ is c-monochromatic. Let n_0 be such that $\lambda(x_{n_0})$ is as small as possible. Comparing $\lambda(x_{n_0})$ with $\lambda(x_{n_0+1})$ and $\lambda(x_{n_0+2})$, we see that either $\lambda(x_{n_0}) < \lambda(x_{n_0+1})$, or $\lambda(x_{n_0+1}) < \lambda(x_{n_0+2})$, or $\lambda(x_{n_0}) = \lambda(x_{n_0+1}) = \lambda(x_{n_0+2})$ which implies $\lambda(x_{n_0}) < \lambda(x_{n_0+1} + x_{n_0+2})$. In either case, we have obtained an element of $\mathrm{AFS}^2(\vec{x})$ whose colour is of the form $(1, j)$; hence all elements of $\mathrm{AFS}^2(\vec{x})$ have the same colour and so λ is strictly increasing on the sequence $\vec{x}$. In particular, this implies that $\lambda(x_n + x_{n+1} + \cdots + x_{n+l}) = \lambda(x_n)$ for all n, l. Therefore, for any n it is possible to take an l large enough that $(0' \cap \lambda(x_n))[\mu(x)] = 0' \cap \lambda(x_n)$ for $x = x_n + x_{n+1} + \cdots + x_{n+l}$; then any element of $\mathrm{AFS}^2(\vec{x})$ having x as its first coordinate will have colour $(1, 1)$. This implies that for every $n \in \mathbb{N}$ we have $(0' \cap \lambda(x_n))[\mu(x_n)] = 0' \cap \lambda(x_n)$ —otherwise, taking any t large enough that $(0' \cap \lambda(x_n))[\mu(y)] = 0' \cap \lambda(x_n)$ for $y = x_{n+1} + \ldots + x_{n+t}$, we would obtain the element $(x_n, y) \in \mathrm{AFS}^2(\vec{x})$ with colour $(1, 0)$—. In other words, the sequence $\vec{x_n}$ is able

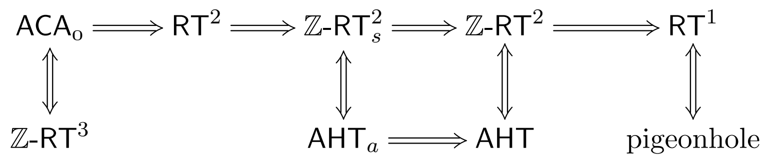

Figure 1. Diagram of implications over $\mathsf{RCA_0}$.

to compute $0'$ (given any $m \in \mathbb{N}$, take an n large enough that $\lambda(x_n) > m$ and check whether m belongs to $(0' \cap \lambda(x_n))[\mu(x_n)]$). □

We summarize the results of the paper, from the perspective of Reverse Mathematics, by presenting the diagram in Figure 1. Whether any of the remaining arrows (except for the one connecting $\mathsf{ACA_0}$ to RT^2) is reversible remains an open question. For example, it is conceivable that AHT already implies AHT_a, although some ideas that have been used for similar results (e.g. [6, Lemma 1]) do not seem to work in this context because the requirement that the sums be adjacent, thus preventing us to skip summands, poses a strong restriction. Similar difficulties arise when one attempts to obtain better lower bounds for $\mathbb{Z}\text{-}\mathsf{RT}^2$: Carlucci [2, Prop. 3] has found that the *Increasing Polarized Ramsey's Theorem for pairs*[2], denoted IPT^2, is a lower bound for AHT_a; the apartness condition plays a crucial rôle in Carlucci's proof, and we have been unable to successfully find a suitable lower bound for AHT without this condition. We do have a conjecture but state it in the form of a question below. A final consideration arises from thinking about versions of the principles we have stated for a bounded number of colours. For example, one could consider, e.g., the principle AHT_k which states the same as AHT but only for colourings with at most k colours. Note that, from this viewpoint, Theorem 14 really proves that $\mathbb{Z}\text{-}\mathsf{RT}^3_4$ implies $\mathsf{ACA_0}$, but it is not clear whether that 4 could be lowered even further. We finalize the paper by stating explicitly these questions.

Questions 15.

(1) *Does* $\mathbb{Z}\text{-}\mathsf{RT}^2$ *(equivalently,* AHT*) imply* $\mathbb{Z}\text{-}\mathsf{RT}^2_s$ *(equivalently,* AHT_a*)?*

(2) *Can one prove, under* $\mathsf{RCA_0}$ *(or possibly under* $\mathsf{RCA_0} + \mathsf{B}\Sigma^0_2$*) that* AHT *(equivalently,* $\mathbb{Z}\text{-}\mathsf{RT}^2$*) imply* D^2*?*

[2]That is, the statement that for every colouring $c : [\mathbb{N}]^2 \longrightarrow k$ there are two infinite sets X, Y such that all pairs of the form $\{x, y\}$ with $x \in X$, $y \in Y$, and $x < y$, have the same colour.

(3) *What are the provable implication relations between the different "bounded colour" versions of* AHT^n_k*, for various* k *(equivalently for* $\mathbb{Z}$-RT^n_k*, for various* k*)?*

The principle D^2 mentioned in point (2) of Questions 15 is the statement, as considered in [3], that for every stable $c : [\mathbb{N}]^2 \longrightarrow k$ (where "stable" means for each n, the function $m \longmapsto c(\{n, m\})$ is eventually constant) there exists an infinite set $X \subseteq \mathbb{N}$ and a colour $i < k$ such that for each $n \in X$, we have $\lim_{m\to\infty} c(\{n, m\}) = i$; this statement is weaker than $\mathsf{IPT}^2 \Rightarrow \mathsf{D}^2$ by a result of Dzhafarov and Hirst [3]. On the other hand, regarding point (3) of Questions 15, it is worth noting that D. Tavernelli [10, Section 2.3] has obtained results showing that the versions of AHT with more colours are not Weihrauch-reducible to the versions with fewer colours, although this still leaves open the possibility of having some version of AHT with more colours being *provable* (over RCA_0, or maybe over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$) from another version with fewer colours.

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