Hindman's theorem as a weak version of the Axiom of Choice

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If $|X| \geq 6$*, and* $c : [X]^2 \longrightarrow 2$*, then there are distinct* $x, y, z \in X$ *such that* $[\{x, y, z\}]^2$ *is* c*-monochromatic.*

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If $|X| \geq 6$, and $c : [X]^2 \longrightarrow 2$, then there are distinct $x, y, z \in X$ such that $[\{x, y, z\}]^2$ is c-monochromatic.

Theorem (Ramsey, 1930)

If X is infinite, and $c : [X]^2 \longrightarrow 2$, then there is an infinite $Y \subseteq X$ such that $[Y]^2$ is monochromatic.

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Theorem (Schur, 1912)

If $c : \mathbb{N} \longrightarrow 2$, then there are distinct $x, y \in \mathbb{N}$ such that $\{x, y, x + y\}$ is monochromatic.

Theorem (Folkman–Rado–Sanders, 1969)

If $c : \mathbb{N} \longrightarrow 2$*, then there are distinct* $x, y, z \in \mathbb{N}$ *such that*
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If $c : \mathbb{N} \longrightarrow 2$, then there are distinct $x_1, \ldots, x_n \in \mathbb{N}$ such that $\mathrm{FS}(x_1, \ldots, x_n) = \{x_{i_1} + \cdots + x_{i_k} \big| i_1 < \ldots < i_k$ and $k \leq n\}$ is c-monochromatic.

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If $c : [\mathbb{N}]^{<\omega} \longrightarrow 2$*, then there is an infinite, pairwise disjoint family* $Y \subseteq [\mathbb{N}]^{<\omega}$ *such that* $\mathrm{FU}(Y) = \left\{F_1 \cup \cdots \cup F_n \middle| n \in \mathbb{N} \text{ and } F_1, \ldots, F_n \in Y\right\}$ *is* c*-monochromatic.*

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The question is, is it really weaker? If so, how much?

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Corollary

In ZF, *the conjunction of* KL *and* HT *is equivalent to Fin=D-Fin.*

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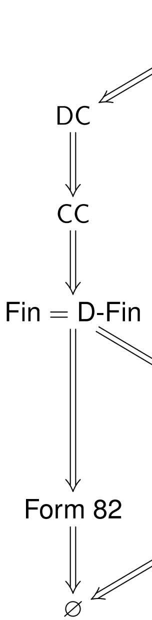
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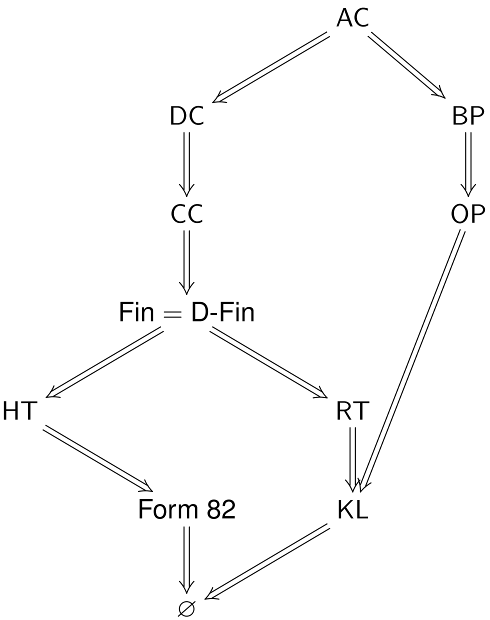
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Equivalently, x is symmetric if there is an $H \in \mathscr{F}$ such that $(\forall \pi \in H)(\pi(x) = x)$.

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Let φ be a "reasonable" statement. If there exists a Fränkel–Mostowski model $M(A, G, \mathscr{F})$ satisfying φ, then there exists a model of ZF $+ \varphi$.

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$\mathsf{ZF} \nvdash \mathsf{RT} \Rightarrow \mathsf{HT}$.

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Proof: Let A be countable, $G = \mathrm{Sym}(A)$, and $\mathscr{F}$ consist of all groups that contain some $G_F = \{\pi \in G \big| \pi \upharpoonright F = \mathrm{Id} \upharpoonright F\}$, with $F \in [A]^{<\omega}$.

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Form 82 holds:

Hindman's theorem fails:

The paper

HINDMAN'S THEOREM IN THE HIERARCHY OF CHOICE PRINCIPLES

DAVID FERNÁNDEZ-BRETÓN

1. Introduction

We deal with various versions of Ramsey's and Hindman's theorem. For notation, RT^n_k means Ramsey's theorem for n-tuples and k colours, and $\mathsf{HT}_n(k)$ means Hindman's theorem for at most n summands and k colours. Simply writing $\mathsf{HT}(k)$ means the full, unrestricted Hindman's theorem for k colours. For a fixed k, it follows[1] from [2, Theorem 3.2] that $\mathsf{HT}(k)$ is equivalent to $\mathsf{HT}_n(k)$ whenever $n \geq 4$, and $\mathsf{HT}(k) \Rightarrow \mathsf{HT}_3(k) \Rightarrow \mathsf{HT}_2(k)$. Also, by [2, Theorem 3.8], we have $\mathsf{RT}^2_k \Rightarrow \mathsf{HT}_2(k)$.

Forster and Truss [3, Lemma 2.2] proved that, for each fixed n, all of the statements RT^n_k are equivalent, and so from now on we will drop the subscript and only refer to the statements RT^n. These authors [3, Theorem 2.3] also establish that, if $n \leq m$, then $\mathsf{RT}^m \Rightarrow \mathsf{RT}^n$. It would be interesting to show that these implications are not reversible (and we will attempt to do so by playing with the Random-hypergraph models). It is worth mentioning that the statement RT^2 appears in [4] as Form 17, whereas the statement $(\forall n)(\mathsf{RT}^n)$ is Form 325.

For the moment, at least we can establish the analog of the aforementioned result (being able to forget colours) for Hindman's theorem. Is there a better result? (One that works even for versions of Hindman's theorem with restricted number of summands.) For a moment there I thought I had it, but now I'm not so sure –so think about this!!!–

Proposition 1. *All of the statements* $\mathsf{HT}(k)$*, as* k *varies, are equivalent.*

Proof. Since k-colourings are always also k'-colourings whenever $k \leq k'$, we have that $\mathsf{HT}(k') \Rightarrow \mathsf{HT}(k)$ under these circumstances. Now to finish the proof, we need only show that $\mathsf{HT}(k) \Rightarrow \mathsf{HT}(k+1)$ for $k \geq 2$ (which yields an argument by induction). So suppose that $k \geq 2$ and that $\mathsf{HT}(k)$ holds. Let X be an infinite set and let $c : [X]^{<\omega} \longrightarrow k+1$ be a colouring. Define another colouring $d : [X]^{<\omega} \longrightarrow k$ by letting $d(x) = \min\{c(x), k-1\}$. Using $\mathsf{HT}(k)$ we obtain an infinite pairwise disjoint family $Y \subseteq [X]^{<\omega}$ such that $\mathrm{FU}(Y)$ is monochromatic for d, say on colour $i < k$. If $i < k-1$ then $\mathrm{FU}(Y)$

[1] In the two references that follow, what was really proved is the case $k = 2$, but it is clear from a cursory reading of the proof that this can be adapted to any k.

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The pre-quel

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Finiteness classes arising from Ramsey-theoretic statements in set theory without choice

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A B S T R A C T

We investigate infinite sets that witness the failure of certain Ramsey-theoretic statements, such as Ramsey's or (appropriately phrased) Hindman's theorem; such sets may exist if one does not assume the Axiom of Choice. We obtain very precise information as to where such sets are located within the hierarchy of infinite Dedekind-finite sets.

1. Introduction

A very interesting line of research in choiceless set theory consists of exploring the relations between the various different ways of expressing finiteness of a set. The starting point for this vein of research is the observation that Dedekind's definition of an infinite set [9, Definition 64, p. 63], which in normal circumstances (i.e. when one assumes the Axiom of Choice, which will henceforth be denoted by AC) is

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Thanks!