

# A CHARACTERIZATION OF THE BOOLEAN PRIME IDEAL THEOREM IN TERMS OF FORCING NOTIONS

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ABSTRACT. For certain weak versions of the Axiom of Choice (most notably, the Boolean Prime Ideal theorem), we obtain equivalent formulations in terms of partial orders, and filter-like objects within them intersecting certain dense sets or antichains. This allows us to prove some consequences of the Boolean Prime Ideal theorem using arguments in the style of those that use Zorn's Lemma, or Martin's Axiom.

## 1. INTRODUCTION

It is well-known that the Axiom of Choice is equivalent to Zorn's Lemma, a statement about the existence of certain elements in some partial orders, which allows us to prove existence results by defining a partial order of approximations to the object whose existence is being established. Very similar in spirit are the so-called forcing axioms, which are combinatorial principles (that typically go beyond the ZFC axioms, while consistent with these, so that they can be used for consistency proofs) that also involve the use of partial orders in their application. In order to state precisely what a forcing axiom is, we will proceed to introduce the necessary definitions.

**Definition 1.1.** Let  $\mathbb{P}$  be a partially ordered set (and denote the corresponding partial order by  $\leq$ ). Then

- (1) We will typically refer to elements of  $\mathbb{P}$  as **conditions**,
- (2) we will say that the condition  $p$  **extends** the condition  $q$  if  $p \leq q$ ,
- (3) we will say that the two conditions  $p, q$  are **compatible**, which we will denote by  $p \not\perp q$ , if they have a common extension, i.e. if there exists a condition  $r$  such that  $r \leq p$  and  $r \leq q$ ,
- (4) we will say that the two conditions  $p, q$  are **incompatible**, denoted  $p \perp q$ , if they are not compatible,
- (5) a subset  $A \subseteq \mathbb{P}$  will be called an **antichain** if any two distinct conditions  $p, q \in A$  must be incompatible,
- (6) we say that a subset  $D \subseteq \mathbb{P}$  is **dense** if every condition has an extension in  $D$ , i.e.  $(\forall p \in \mathbb{P})(\exists q \in D)(q \leq p)$ ,
- (7) a subset  $G \subseteq \mathbb{P}$  will be called a **filter** if it is closed upwards (this is, if  $(\forall p \in G)(\forall q \in \mathbb{P})(p \leq q \Rightarrow q \in G)$ ), and for every  $p, q \in G$  there exists an  $r \in G$  which extends both  $p$  and  $q$ ,
- (8) if  $\mathcal{D}$  is a family of dense subsets of  $\mathbb{P}$  (respectively, if  $\mathcal{A}$  is a family of antichains) then the filter  $G$  will be called  **$\mathcal{D}$ -generic** if it intersects every element of  $\mathcal{D}$  (respectively,  **$\mathcal{A}$ -generic** if it intersects every element of  $\mathcal{A}$ ).

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The previous definition contains all the terminology needed to talk about forcing axioms. A partial order  $\mathbb{P}$  is said to be **c.c.c.** if every antichain is countable. Historically, the first example of a forcing axiom would be the one that is known as Martin's Axiom<sup>1</sup>, abbreviated MA.

**MA** For every c.c.c. partial order  $\mathbb{P}$  and every family  $\mathcal{D}$  of dense sets with  $|\mathcal{D}| < \mathfrak{c}$ , there exists a  $\mathcal{D}$ -generic filter.

Other variations of this combinatorial principle have been proposed over the years, and this has eventually led to the formulation of an abstract template for a forcing axiom, which is as follows: let  $\mathcal{X}$  be a class of partial orders, and  $\kappa$  a cardinal. Then we can state the forcing axiom

**FA $_{\kappa}(\mathcal{X})$**  For every partial order  $\mathbb{P} \in \mathcal{X}$  and every family  $\mathcal{D}$  of dense sets satisfying  $|\mathcal{D}| \leq \kappa$ , there exists a  $\mathcal{D}$ -generic filter.

Not all forcing axioms necessarily fall neatly into this template, so sometimes we will state some combinatorial principles that deviate slightly from it. For instance, we might want to use something other than a cardinal number in the place of the subindex in our template. An obvious example would be **FA $_{\infty}(\mathcal{X})$** , which should be interpreted as the statement that for every partial order  $\mathbb{P} \in \mathcal{X}$  and every family  $\mathcal{D}$  of dense sets (where  $\mathcal{D}$  can be completely arbitrary, without restrictions of cardinality or of any other property), there exists a  $\mathcal{D}$ -generic filter. Similarly, expressions such as **FA $_{<\kappa}(\mathcal{X})$**  should be given the obvious meaning.

This notation allows us to state several forcing axioms in a compact way; for example MA becomes the statement **FA $_{<\mathfrak{c}}$ (c.c.c.)**, the combinatorial principle  $\mathfrak{t} = \mathfrak{c}$  is equivalent (by Bell's Theorem [2], together with Malliaris and Shelah's [9, 10] recent result that  $\mathfrak{p} = \mathfrak{t}$ ) to **FA $_{<\mathfrak{c}}$ ( $\sigma$ -centred)**<sup>2</sup>, and the combinatorial principle  $\text{cov}(\mathcal{M}) = \mathfrak{c}$  is equivalent to **FA $_{<\mathfrak{c}}$ (countable)**. The Proper Forcing Axiom is, of course, the statement **FA $_{\omega_1}$ (proper)**<sup>3</sup>.

Although Zorn's Lemma is a statement that concerns partial orders, at first sight it does not look like a forcing axiom, for the object whose existence it asserts is not a filter. One of the first results in this paper shows that this first impression is misguided, and it is possible to rephrase Zorn's Lemma as a perfectly legitimate forcing axiom. This continues a certain line of research, that has been pursued [15] in the past, concerning the possibility of expressing the Axiom of Choice, or some weak versions of it, as forcing axioms. The most immediate example of this is the following result<sup>4</sup>. Recall that the principle of Dependent Choice, abbreviated DC, is the statement that, for every set  $X$  equipped with a binary relation  $R$  satisfying  $(\forall x \in X)(\exists y \in X)(x R y)$ , there exists a sequence  $\langle x_n \mid n < \omega \rangle$  such that  $(\forall n < \omega)(x_n R x_{n+1})$ .

**Theorem 1.2** (Todorćević). *In the theory ZF, the following two statements are equivalent:*

- (1) DC
- (2) **FA $_{\omega}(\mathcal{P})$** , where  $\mathcal{P}$  is the class of all partial orders.

The interested reader can find a full proof of Theorem 1.2, as well as of Theorem 1.3 below, in [12, Theorem 3.2.4]. The next theorem that we mention characterizes the full Axiom of Choice, abbreviated AC, as a collection of forcing axioms. Recall that, for a

<sup>1</sup>For an introduction to Martin's Axiom and its consequences, see [7, Chapter II.2].

<sup>2</sup>Recall that a partial order is  $\sigma$ -centred if it can be written as the union of countably many filters.

<sup>3</sup>We omit the definition of a proper partial order, since it will not be relevant for this paper, but direct the interested reader to [16] or [1].

<sup>4</sup>This result is attributed to Todorćević, although Goldblatt [4] came quite close to stating it.

cardinal number  $\kappa$ , a partial order  $\mathbb{P}$  is said to be  **$\kappa$ -closed** if every descending sequence  $\langle p_\alpha \mid \alpha < \lambda \rangle$  of conditions of length  $\lambda < \kappa$  has a lower bound<sup>5</sup>.

**Theorem 1.3** (Todorćević). *In the theory ZF, the following two statements are equivalent:*

- (1) AC
- (2) for every cardinal  $\kappa$ ,  $\text{FA}_\kappa$  ( $\kappa$ -closed).

Other results in this line of research have been found by Gary Shannon [15], who found characterizations of König's Lemma and of the principle of Countable Choice as forcing axioms.

The second section of this paper contains yet another characterization of AC as a forcing axiom. This characterization consists of explaining how Zorn's Lemma can be rephrased so that it looks like a legitimate forcing axiom. This technique can also be used to obtain another characterization of DC as well, and we also improve Shannon's characterization of König's Lemma [15, Theorem 2]. We further provide a characterization of a further weak choice principle, which is in the spirit of [15, Corollary 2]. Then in the third section, we prove what we consider to be the main result of this paper: we characterize the Boolean Prime Ideal theorem in terms of a statement that is very close to a forcing axiom. This statement allows to prove consequences of the Boolean Prime Ideal theorem by using the same type of reasonings that any forcing axiom allows, and show three examples of this.

## 2. AXIOM OF CHOICE, KÖNIG'S LEMMA, AND LINEARLY ORDERED SETS

In this section, we provide some characterizations of certain weak principles of choice, including the Axiom of Choice itself, as forcing axioms. We first introduce a definition that meshes together the partial orders that are typically used in forcing axioms, with those that concern Zorn's Lemma.

**Definition 2.1.** We will say that a partial order  $\mathbb{P}$  is **semi-separative** if for every  $p \in \mathbb{P}$ , either  $p$  is minimal or  $p$  has two incompatible extensions in  $\mathbb{P}$ .

Together with the previous definition, the following lemma will be very useful for our characterization of AC.

**Lemma 2.2.** *Let  $\mathbb{P}$  be any semi-separative partial order, and  $G \subseteq \mathbb{P}$ . Then  $G$  intersects every dense set in  $\mathbb{P}$  if and only if  $G = \{q \in \mathbb{P} \mid p \leq q\}$  for some minimal element  $p$ .*

*Proof.* Suppose first that  $G = \{q \in \mathbb{P} \mid p \leq q\}$  for some minimal element  $p$ , in particular  $p \in G$ . Note that, since  $p$  is minimal, then  $p \in D$  for every dense  $D \subseteq \mathbb{P}$ . Hence  $G$  intersects every dense set.

Conversely, suppose that  $G$  is a filter meeting every dense subset  $D \subseteq \mathbb{P}$ . Therefore the set  $\mathbb{P} \setminus G$  cannot be dense, for it does not meet  $G$ . So there is an element  $p \in G$  such that every extension of  $p$  is an element of  $G$ . Thus, it cannot be the case that  $p$  has two incompatible extensions, since any two elements of  $G$  must be compatible. Since  $\mathbb{P}$  is semi-separative,  $p$  must then be minimal. We now claim that  $G = \{q \in \mathbb{P} \mid p \leq q\}$ . Since  $p \in G$ , clearly  $\{q \in \mathbb{P} \mid p \leq q\} \subseteq G$ , now to prove the converse inclusion, let  $q \in G$ . Since  $G$  is a filter, there exists an  $r \in G$  that extends both  $p$  and  $q$ , so by minimality of  $p$  we obtain that  $r = p$  and so  $p \leq q$  for every  $q \in G$ .  $\square$

**Definition 2.3.** We will say that a partial order  $\mathbb{P}$  is a **Zorn partial order** if every linearly ordered subset of  $\mathbb{P}$  has a lower bound.

<sup>5</sup>With this definition, every partial order is  $\omega$ -closed, but being  $\kappa$ -closed for some  $\kappa \geq \omega_1$  is a nontrivial condition to impose on a partial order.

Lemma 2.2 promptly allows us to prove the following characterization of the Axiom of Choice. Recall that Zorn’s Lemma, which is equivalent to AC, is the statement that every Zorn partial order has a minimal element (it is usually stated in terms of upper bounds and maximal elements, but of course both versions are equivalent).

**Theorem 2.4.** *In the theory ZF, the following two statements are equivalent:*

- (1) AC,
- (2)  $\text{FA}_\infty(\mathcal{L})$ , where  $\mathcal{L}$  is the class of all semi-separative Zorn partial orders.

*Proof.*

1  $\Rightarrow$  2: Since AC implies Zorn’s Lemma, which asserts that every Zorn partial order has a minimal element, in particular every semi-separative Zorn partial order  $\mathbb{P}$  has a minimal element. By Lemma 2.2, this minimal element gives rise to a fully generic (i.e.  $\mathcal{D}$ -generic where  $\mathcal{D}$  is the collection of all dense subsets of  $\mathbb{P}$ ) filter  $G \subseteq \mathbb{P}$ .

2  $\Rightarrow$  1: The fact that Zorn’s Lemma implies AC, together with Lemma 2.2, should finish the proof, but we will be a bit more explicit. So let  $X$  be a family of nonempty sets. If  $X_1 = \{x \in X \mid |x| = 1\}$  and  $X_2 = X \setminus X_1$ , and  $f$  is any choice function on  $X_2$ , then  $f \cup \{\langle x, \bigcup x \rangle \mid x \in X_1\}$  will be a choice function on  $X$ , hence we assume without loss of generality that every element of  $X$  has at least two elements. Now let

$$\mathbb{P} = \{f : Y \longrightarrow \bigcup X \mid Y \subseteq X \wedge f \text{ is a choice function on } Y\},$$

with the order given by  $f \leq g$  iff  $f \supseteq g$ . It is easy to see that  $\mathbb{P}$  is a semi-separative Zorn partial order (the fact that non-minimal elements have two incompatible extensions follows from our assumption that every  $x \in X$  has at least two elements), so by hypothesis there exists a generic filter  $G \subseteq \mathbb{P}$ , and clearly letting  $h = \bigcup G$  will yield that  $h : X \longrightarrow \bigcup X$  is a choice function on  $X$  (the fact that  $X = \text{dom}(h)$  follows from the fact that  $D_x = \{f \in \mathbb{P} \mid x \in \text{dom}(f)\}$  is dense for every  $x \in X$ ).

□

We will omit the proof of the following theorem, since it is completely analogous to that of Theorem 2.4, once one remembers that DC is equivalent (under ZF) to the statement that every partial order  $\mathbb{P}$  such that every linearly ordered subset  $X \subseteq \mathbb{P}$  is finite, has a minimal element<sup>6</sup>.

**Theorem 2.5.** *In the theory ZF, the following two statements are equivalent:*

- (1) DC,
- (2)  $\text{FA}_\infty(\mathcal{F})$ , where  $\mathcal{F}$  is the class of all semi-separative partial orders such that every linearly ordered subset is finite.

We now introduce some notation regarding weak choice principles. The symbol  $\text{AC}(\kappa, \lambda)$  denotes the statement that every family of cardinality at most  $\kappa$ , each of whose elements is nonempty and has cardinality at most  $\lambda$ , has a choice function. Variations of this notation, where instead of a cardinal we have something like WO which stands for well-orderable, should be given the obvious meaning. Our next theorem is, in fact, a small improvement over a theorem of Shannon [15, Theorem 2], who showed that König’s Lemma is equivalent to a forcing axiom, for  $\omega$  many dense sets, over a certain class of partial orders with a quite complex definition. The authors noticed that the description of the class of partial

<sup>6</sup>This statement is referred to as “Form 43L” in [5, p. 31].

orders involved could be made much simpler by removing an unnecessary condition whose use in the corresponding proof can be skipped.

**Theorem 2.6.** *In the theory ZF, the following four statements are equivalent:*

- (1)  $\text{AC}(\omega, < \omega)$ ,
- (2) König's Lemma,
- (3) every countable union of finite sets is countable,
- (4)  $\text{FA}_\omega(\mathcal{C})$ , where  $\mathcal{C}$  is the class of all partial orders whose underlying set is a countable union of finite sets.

*Proof.*

$1 \Leftrightarrow 2 \Leftrightarrow 3$ : These equivalences are well-known (see e.g. [5, pp.19–20]).

$3 \Rightarrow 4$ : If  $\mathbb{P} \in \mathcal{C}$ , then our assumption implies that  $\mathbb{P}$  is countable, and in particular well-orderable. Hence we have a choice function on the powerset of  $\mathbb{P}$ , so whenever we have a sequence  $\langle D_n \mid n < \omega \rangle$  of countably many dense sets, we can use the aforementioned choice function to recursively define a decreasing sequence  $\langle p_n \mid n < \omega \rangle$  such that  $p_n \in D_n$ . Clearly  $G = \{q \in \mathbb{P} \mid (\exists n < \omega)(p_n \leq q)\}$  will be a  $\{D_n \mid n < \omega\}$ -generic filter.

$4 \Rightarrow 1$ : Let  $\{X_n \mid n < \omega\}$  be a countable family of nonempty finite sets, indexed by  $\omega$ . We let

$$\mathbb{P} = \left\{ f : n \longrightarrow \bigcup_{k < n} X_k \mid n < \omega \wedge f \text{ is a choice function on } \{X_k \mid k < n\} \right\}.$$

It can be easily verified that the partially ordered set  $\mathbb{P}$  is the countable union of the finite sets  $F_n = \{f \in \mathbb{P} \mid \text{dom}(f) = n\}$ . Hence if we consider, for every  $n < \omega$ , the dense set  $D_n = \{p \in \mathbb{P} \mid n \in \text{dom}(p)\}$ , then we obtain a  $\{D_n \mid n < \omega\}$ -generic set  $G$ . It is straightforward to verify that  $\bigcup G$  is a choice function on  $\{X_n \mid n < \omega\}$ .  $\square$

The last result of this section is very much in the spirit of [15, Corollary 2]. This result consists of a characterization of the principle  $\text{AC}(\text{LO}, < \omega)$ , which asserts the existence of choice functions for every linearly orderable collection of nonempty finite sets. This weak choice principle is implied both by the Ordering Principle (asserting that every set can be linearly ordered), and by  $\text{AC}(\infty, < \omega)$  (which asserts the existence of a choice function on any arbitrary family of nonempty finite sets); and it implies  $\text{AC}(\text{WO}, < \omega)$  (asserting the existence of a choice function on any well-orderable family of nonempty finite sets); and none of these implications is reversible [17, Corollary 4.6].

In order to state the equivalence of this weak choice principle with something that resembles a forcing axiom, we will need to be more flexible with our notion of forcing axiom, and consider preorders instead of partial orders. Given  $n < \omega$ , we will denote by  $\mathcal{L}_n$  the class of all preorders whose underlying set is the union of a pairwise disjoint linearly orderable family of finite sets, such that every antichain has size at most  $n$ . We will also denote  $\mathcal{L} = \bigcup_{n < \omega} \mathcal{L}_n$ . Also, a superscript LO in the statement of a forcing axiom will denote the additional assertion that the relevant filter can be taken to be linearly orderable.

**Theorem 2.7.** *Under the theory ZF, the following four statements are equivalent:*

- (1)  $\text{AC}(\text{LO}, < \omega)$ ,
- (2) the union of a pairwise disjoint linearly orderable family of finite sets is linearly orderable,
- (3)  $\text{FA}_\infty^{\text{LO}}(\mathcal{L})$ ,

(4) there exists an  $n$ ,  $1 \leq n < \omega$ , such that  $\text{FA}_\infty^{\text{LO}}(\mathcal{L}_n)$  holds.

*Proof.*

1  $\Rightarrow$  2: Let  $X$  be a pairwise disjoint set, linearly ordered by  $\leq$ , each of whose elements is finite nonempty. Assuming  $\text{AC}(\text{LO}, < \omega)$  we will proceed to construct a linear order  $\preceq$  on  $\bigcup X$ . For each  $x \in X$ , define  $L_x = \{L \subseteq x \times x \mid L \text{ is a linear order on } x\}$ . Then each element in the linearly orderable family  $L = \{L_x \mid x \in X\}$  is finite nonempty, so by  $\text{AC}(\text{LO}, < \omega)$  it is possible to obtain a choice function  $f$  on  $L$ . This allows us to define the relation

$$y \preceq z \iff x_y \leq x_z \text{ or } (x_y = x_z \text{ and } y f(x_y) z) \text{ or } y = z,$$

(where  $x_y$  denotes the unique  $x \in X$  such that  $y \in x$ ) on  $\bigcup X$ . It is straightforward to check that  $\preceq$  linearly orders  $\bigcup X$ .

2  $\Rightarrow$  3: Let  $\mathbb{P} \in \mathcal{L}$ . The first thing to notice, is that by assumption there exists a linear order  $L$  on  $\mathbb{P}$ . Since  $\mathbb{P} \in \mathcal{L}_n$  for some  $n < \omega$ , in particular the size of the antichains of  $\mathbb{P}$  is bounded above. Thus we can take an antichain  $A \subseteq \mathbb{P}$  of maximum possible cardinality. Take any  $p \in A$ , and define  $G_p = \{q \mid q \not\leq p\}$ .  $G_p$  is linearly orderable by  $L \upharpoonright G_p$ , and we claim that  $G_p$  is also a generic filter. If  $D$  is any dense set, there exists a  $q \in D$  with  $q \leq p$ , so  $q \in G_p$  by definition and hence  $G_p$  meets  $D$ . Now to see that  $G_p$  is a filter, let  $q, r \in G_p$ . Then by definition both  $q$  and  $r$  are compatible with  $p$ , so there are extensions  $q' \leq q$  and  $r' \leq r$  that extend  $p$  as well. If  $q'$  and  $r'$  were incompatible, the set  $(A \cup \{q', r'\}) \setminus \{p\}$  would be an antichain of cardinality strictly larger than  $|A|$ , contradicting that  $A$  has maximum possible cardinality. Therefore  $q' \leq r'$  and so  $q \leq r$ . Since  $D = \{s \in \mathbb{P} \mid s \perp q \vee s \perp r \vee (s \leq q \wedge s \leq r)\}$  is a dense set, and we already showed that  $G_p$  must intersect every dense set, and that any two elements in  $G_p$  must be compatible, we can conclude that there exists an  $s \in G_p$  such that  $s \leq q$  and  $s \leq r$ . Thus  $G_p$  is a generic linearly orderable filter.

3  $\Rightarrow$  4: This is immediate.

4  $\Rightarrow$  1: Let  $X$  be a pairwise disjoint, linearly orderable family of nonempty finite sets. We preorder  $\mathbb{P} = \bigcup X$  with the somewhat trivial preorder that makes  $p \leq q$  for every  $p, q \in \mathbb{P}$ . Then every antichain in  $\mathbb{P}$  is a singleton, so  $\mathbb{P} \in \mathcal{L}_n$ . Since every  $x \in X$  is a dense set, when considered as a subset of  $\mathbb{P}$ , our hypothesis gives us a filter  $G \subseteq \mathbb{P}$ , meeting every  $x \in X$ , and equipped with a linear order  $L$ . Hence we can define  $z_x = \min_L(G \cap x)$  for every  $x \in X$  (it is possible to take  $L$ -minima because each  $G \cap x$  is finite and nonempty), now the family  $\{z_x \mid x \in X\}$  will be a selector for  $X$ .

□

### 3. A NEW CHARACTERIZATION OF THE BPI

The Boolean Prime Ideal Theorem, denoted by BPI, is well-known as a statement weaker than AC that still suffices to carry out many of the proofs that require AC. In this section we will prove that the BPI is equivalent to a certain statement that deals with partially ordered sets and the existence of certain filter-like families therein. Afterwards we will show how this statement allows us to prove some consequences of BPI by using the same sort of ideas as in a proof that uses Zorn's Lemma, or Martin's Axiom. We will lay down a couple of definitions that we will need in order to state the version of the BPI that we will use for our proof.

**Definition 3.1.** Let  $X$  be a nonempty set. A family  $M$  consisting of functions from finite subsets of  $X$  into  $2$  will be called a **binary mess** on  $X$  provided that it satisfies

- (1)  $M$  is closed under restrictions, this is, if  $s \in M$  and  $F \subseteq \text{dom}(s)$  then  $s \upharpoonright F \in M$ , and
- (2) for every finite  $F \subseteq X$ , there exists an  $s \in M$  such that  $\text{dom}(s) = F$ .

**Definition 3.2.** Let  $X$  be a nonempty set, and let  $M$  be a binary mess on  $X$ . We will say that a function  $f : X \rightarrow 2$  is **consistent** with  $M$  if for every finite  $F \subseteq X$ ,  $f \upharpoonright F \in M$ .

These definitions allow us to state the following weak choice principle, which is equivalent to the BPI [6, Theorem 2.2].

**The Consistency Principle** For every nonempty set  $X$ , and every binary mess  $M$  on  $X$ , there exists a function  $f : X \rightarrow 2$  that is consistent with  $M$ .

In our search for an equivalent of BPI, we were not able to find a statement which falls neatly into the template that we have been using for a forcing axiom. Inspired by Cowen's generalization of König's Lemma [3, Theorems 1 and 8], we were able to put together the equivalence in Theorem 3.4 below, though we first state a couple of definitions.

**Definition 3.3.** Let  $\mathbb{P}$  be a partially ordered set.

- (1) We will say that a subset  $G \subseteq \mathbb{P}$  is **2-linked** if every two elements of  $G$  are compatible.
- (2) We will say that a family  $\mathcal{A}$  of antichains of  $\mathbb{P}$  is **centred** if for every choice of finitely many antichains  $A_1, \dots, A_n \in \mathcal{A}$ , there exist elements  $p \in \mathbb{P}$  and  $a_1 \in A_1, \dots, a_n \in A_n$  such that  $p \leq a_k$  for all  $1 \leq k \leq n$ .

Under ZFC, every dense set gives rise to a maximal antichain, and viceversa. Additionally, the key property of filters is not so much that they are closed upwards, or that any two of its elements have a common extension in the filter itself, but rather, just that any two of its elements are compatible. This is what allows us to properly glue together the elements of a filter in order to obtain the desired object, in most of the proofs that use forcing axioms. Therefore we claim that the characterization in Theorem 3.4 below does not deviate excessively from the usual template of a forcing axiom.

**Theorem 3.4.** *The theory ZF proves that the following two statements are equivalent:*

- (1) BPI,
- (2) for every partial order  $\mathbb{P}$ , and every centred family  $\mathcal{A}$  of finite antichains, there exists an  $\mathcal{A}$ -generic 2-linked subset  $G \subseteq \mathbb{P}$ .

*Proof.*

$1 \Rightarrow 2$ : Assume BPI, which is equivalent to the Consistency Principle, and let  $\mathbb{P}$  be a partial order, with a centred family  $\mathcal{A}$  of finite antichains. Define a binary mess  $M$  on  $\mathbb{P}$  by

$$M = \left\{ m : F \rightarrow 2 \mid F \in [\mathbb{P}]^{<\omega} \wedge (\forall x, y \in F)(m(x) = m(y) = 1 \Rightarrow x \not\leq y) \wedge (\forall A \in \mathcal{A})(A \subseteq F \Rightarrow (\exists x \in A)(m(x) = 1)) \right\}.$$

It is readily checked that  $M$  is closed under restrictions. Now if  $F \subseteq \mathbb{P}$  is finite, then there are only finitely many elements  $A \in \mathcal{A}$  with  $A \subseteq F$ , let those be  $A_1, \dots, A_n$ . Since  $\mathcal{A}$  is centred, there exists a  $p \in \mathbb{P}$  and  $a_k \in A_k$  for each  $1 \leq k \leq n$  such that  $p \leq a_k$ . We define  $m : F \rightarrow 2$  by  $m(x) = 1 \iff (\exists k \leq n)(x = a_k)$ . It is

readily checked that  $m \in M$ , thus  $M$  is indeed a binary mess. So the Consistency Principle ensures the existence of a function  $f : \mathbb{P} \rightarrow 2$  consistent with  $M$ . Letting  $G = \{x \in \mathbb{P} \mid f(x) = 1\}$  will give us an  $\mathcal{A}$ -generic 2-linked set.

$2 \Rightarrow 1$ : Let  $M$  be a binary mess on some nonempty set  $X$ . We partially order  $M$  itself by reverse inclusion, and for each finite  $F \subseteq X$  we let  $A_F = \{m \in M \mid \text{dom}(m) = F\}$ . Then each of the  $A_F$  is a finite antichain (in fact,  $|A_F| \leq 2^{|F|}$ ). We define  $\mathcal{A} = \{A_F \mid F \in [X]^{<\omega}\}$ , and proceed to verify that  $\mathcal{A}$  is indeed a centred family of antichains, so let  $A_{F_1}, \dots, A_{F_n} \in \mathcal{A}$ . Since  $M$  is a binary mess, there exists an  $m \in M$  with  $\text{dom}(m) = F_1 \cup \dots \cup F_n$ . For each  $1 \leq k \leq n$ , we have that  $m_k = m \upharpoonright F_k \in A_{F_k}$  is an element extended by  $m$ , thus  $\mathcal{A}$  is a centred family. Therefore, by assumption there exists an  $\mathcal{A}$ -generic 2-linked family  $G$ .

Now we claim that defining  $f = \bigcup G$  yields a function consistent with  $M$ . Note first that  $f$  must be a function, since  $G$  is linked. Furthermore, for each  $x \in X$ ,  $G$  must meet  $A_{\{x\}}$ , which implies that  $x \in \text{dom}(f)$ . Hence  $\text{dom}(f) = X$ . Lastly,  $f$  is consistent with  $M$  because, for each finite  $F \subseteq X$ ,  $G$  must intersect  $A_F$ , and it is easy to see that  $G \cap A_F$  consists of the single element  $f \upharpoonright F \in M$ . □

We now present three examples of proofs using the equivalence found in Theorem 3.4, in order to illustrate how we can use this new equivalence to prove consequences of BPI in the spirit of proofs that use forcing axioms. Our first example is the proof of the Ordering Principle.

**Example 3.5.** The ordering principle is the statement that every set can be linearly ordered. This principle is implied by the BPI, and the implication is not reversible<sup>7</sup>. Thus we will prove the ordering principle, in ZF, under the assumption that statement (2) in Theorem 3.4 holds.

So let  $X$  be an arbitrary (nonempty) set. Partially order the set

$$\mathbb{P} = \{L \mid L \text{ is a linear order, and } \text{dom}(L) \in [X]^{<\omega}\}$$

by reverse inclusion, so that  $L \leq L'$  iff  $L \supseteq L'$ . For each  $F \in [X]^{<\omega}$  we let  $A_F$  be the collection of all linear orders on  $F$ , which is a finite (in fact, of size  $|F|!$ ) antichain in  $\mathbb{P}$ . So the family  $\mathcal{A} = \{A_F \mid F \in [X]^{<\omega}\}$  consists of finite antichains; we will now proceed to prove that it is centred, so consider finitely many elements  $A_{F_1}, \dots, A_{F_n} \in \mathcal{A}$ . Let  $L$  be a linear order on the finite set  $F_1 \cup \dots \cup F_n$ , then  $L$  simultaneously extends each of the elements  $L \upharpoonright F_k \in A_{F_k}$ , for  $1 \leq k \leq n$ , and therefore  $\mathcal{A}$  is linked. Thus we obtain an  $\mathcal{A}$ -generic 2-linked set  $G$ . It is readily checked that  $\bigcup G$  is, in fact, a linear order on  $X$ .

**Example 3.6.** We shall now consider a statement which is actually known to be equivalent to the BPI. We will show only half of the equivalence, the half that illustrates the use of our new characterization of the BPI.

Let  $\{A_i \mid i \in I\}$  be a collection of finite sets, and let  $S$  be a symmetric binary relation on  $A = \bigcup_{i \in I} A_i$ . We will say that a function  $f$  with range contained in  $A$  is  $S$ -consistent if  $(\forall x, y \in \text{dom}(f))(f(x) S f(y))$ . The statement that we will prove, assuming BPI, is the following: If for every finite  $W \subset I$  there is an  $S$ -consistent choice function for  $\{A_i \mid i \in W\}$ , then there is an  $S$ -consistent choice function for the whole family  $\{A_i \mid i \in I\}$ .<sup>8</sup>

<sup>7</sup>Mathias [11] proved that the Ordering Principle does not imply the Order Extension Principle (the statement that every partial ordering can be extended to a total ordering), which is another consequence of BPI.

<sup>8</sup>The fact that the BPI is equivalent to this statement for all collections of finite sets  $\{A_i \mid i \in I\}$  and all symmetric relations  $S$  on  $\bigcup_{i \in I} A_i$  is proved in [8, Theorem 2\*].

For this, we let

$$\begin{aligned} \mathbb{P} = \{ & p \mid p : W \longrightarrow A \text{ for some finite } W \subseteq I \text{ and} \\ & p \text{ is an } S\text{-consistent choice function on } \{A_i \mid i \in W\}\}, \end{aligned}$$

and we partially order  $\mathbb{P}$  by reverse extension (i.e.  $p \leq q$  iff  $p \supseteq q$ ). For each finite  $W \subseteq I$ , the family  $A_W = \{p \in \mathbb{P} \mid \text{dom}(p) = W\}$  is clearly a finite antichain, and for each finite collection of these antichains,  $A_{W_1}, \dots, A_{W_n}$ , it is clear that any  $S$ -consistent choice function  $p$  on  $\{A_i \mid i \in W_1 \cup \dots \cup W_n\}$  (there exists at least one by hypothesis) will extend the elements  $p \upharpoonright W_i \in A_{W_i}$  for every  $1 \leq i \leq n$ . Therefore, letting  $\mathcal{A} = \{A_W \mid W \in [I]^{<\omega}\}$  yields a centred family of finite antichains. Thus by the assumption that BPI holds and Theorem 3.4, we can obtain an  $\mathcal{A}$ -generic 2-linked family  $G$ . We claim that  $f = \bigcup G$  is an  $S$ -consistent choice function on  $\{A_i \mid i \in I\}$ . It is easy to see that  $f$  is a function because  $G$  is 2-linked. Moreover,  $\text{dom}(f) = I$  since  $G$  must intersect each of the  $A_{\{i\}}$ , for all  $i \in I$ . Finally, given any two  $i, j \in I$ , we can derive from the fact that  $G$  meets  $A_{\{i,j\}}$  that  $f \upharpoonright \{i, j\} \in G \subseteq \mathbb{P}$ , and therefore we must have that  $f(i) S f(j)$ . Hence  $f$  is  $S$ -consistent, and we are done.

Our last example will be a proof of the Hahn-Banach theorem, which is an extremely well-known result. This theorem is implied by the BPI [8, Section 5], though the implication is not reversible [14].

**Example 3.7.** We will prove the Hahn-Banach theorem from statement (2) in Theorem 3.4, so we will need to introduce some terminology. If  $V$  is a real vector space, a **Minkowski functional** on  $V$  is a function  $p : V \longrightarrow \mathbb{R}$  satisfying  $p(x+y) \leq p(x) + p(y)$  and  $p(tx) = tp(x)$  for every  $x, y \in V$  and every positive  $t \in \mathbb{R}$ . A **linear functional** is simply a linear transformation  $f : V \longrightarrow \mathbb{R}$ . The Hahn-Banach theorem is the following statement: for every real vector space  $V$ , for every Minkowski functional  $p : V \longrightarrow \mathbb{R}$  on  $V$  and every linear functional  $f : W \longrightarrow \mathbb{R}$  defined on a subspace  $W$  of  $V$ , such that  $(\forall x \in W)(f(x) \leq p(x))$ , there exists a linear functional  $\hat{f} : V \longrightarrow \mathbb{R}$  extending  $f$  such that  $(\forall x \in V)(\hat{f}(x) \leq p(x))$ .

In order to carry out our proof, we would like to define a partial order of approximations to the desired functional, but doing this carelessly gets us at risk of not being able to obtain small enough antichains. So our approximations will consist on *approximating* (though not specifying) the value of the desired functional on finitely many vectors, for which we need to deal with certain special kinds of intervals. Given a closed interval  $I = [a, b] \subseteq \mathbb{R}$  and  $n < \omega$ , we will define a **dyadic subinterval of depth  $n$**  of  $I$  to be any interval of the form  $\left[ a + \frac{(b-a)k}{2^n}, a + \frac{(b-a)(k+1)}{2^n} \right]$  where  $k \in \{0, 1, \dots, 2^n - 1\}$ . A **dyadic subinterval** of  $I$  will be a dyadic subinterval of depth some  $n < \omega$ ; we denote by  $I_n$  the collection of all ( $2^n$  many) dyadic subintervals of depth  $n$  of  $I$ , and by  $I_\infty = \bigcup_{n < \omega} I_n$  the collection of all dyadic subintervals of  $I$  (of any depth). Dealing with dyadic subintervals will be very useful because there are only finitely many dyadic subintervals of any given depth, and because, given any closed interval  $I$ , any two of its dyadic subintervals  $J, J' \in I_\infty$  will satisfy that either  $|J \cap J'| \leq 1$ , or  $J \subseteq J'$  or  $J' \subseteq J$ .

We can now finally proceed to the proof. Assume the corresponding hypotheses, i.e. let  $V$  be a linear vector space,  $p : V \longrightarrow \mathbb{R}$  a Minkowski functional, and  $f : W \longrightarrow \mathbb{R}$  be a linear functional, defined on the (proper) subspace  $W$  of  $V$ , such that  $(\forall x \in W)(f(x) \leq p(x))$ . For each  $x \in V$  we define the closed interval  $I(x) = [-p(-x), p(x)]$ . Note that if we manage to define  $\hat{f}$  in such a way that  $\hat{f}(x) \in I(x)$  for every  $x$ , we will have taken care of the inequality requirement for  $\hat{f}$  (and hence we would only need to work towards ensuring that

$\hat{f}$  is linear). Thus we define our partially ordered set  $\mathbb{P}$  to consist of all functions  $g$  with domain some finite  $X \subseteq V$  that satisfy

- (1)  $(\forall x \in X)(g(x) \in I(x)_\infty)$ ,
- (2)  $(\forall x \in X)(x \in W \Rightarrow f(x) \in g(x))$ ,
- (3)  $(\forall x, y \in X)(x + y \in X \Rightarrow (g(x) + g(y)) \cap g(x + y) \neq \emptyset)$ , and
- (4)  $(\forall x \in X)(\forall r \in \mathbb{R})(rx \in X \Rightarrow rg(x) \cap g(rx) \neq \emptyset)$ .

Hence every  $g \in \mathbb{P}$  specifies, for a finite number of  $x \in V$ , a dyadic subinterval of  $I(x)$  from which we intend to eventually pick the value  $\hat{f}(x)$ , in a way that is consistent with the fact that we want  $\hat{f}$  to be a linear functional extending  $f$ . The partial ordering is given by:  $g \leq g'$  iff  $\text{dom}(g') \subseteq \text{dom}(g)$  and  $(\forall x \in \text{dom}(g'))(g(x) \subseteq g'(x))$ . For each function  $h : X \rightarrow \omega$ , with domain some  $X \in [V]^{<\omega}$ , we let  $A_h = \{g \in \mathbb{P} \mid \text{dom}(g) = X \text{ and } (\forall x \in X)(g(x) \in I(x)_{h(x)})\}$ .

**Claim 3.8.** *Each of the  $A_h$  is a nonempty finite antichain.*

*Proof of Claim 3.8.* The fact that  $A_h$  is a finite antichain follows directly from the fact that the interval  $I(x)$  has only finitely many dyadic subintervals of depth  $h(x)$ , for each of the finitely many  $x \in \text{dom}(h)$  (and that the intersection of any two dyadic intervals of the same depth is either empty or a singleton). The nontrivial part of the claim is thus the nonemptiness of  $A_h$ . For this, we will use the following fact (which we will not prove because it properly belongs to Functional Analysis rather than Set Theory): for every linear functional  $l : W' \rightarrow \mathbb{R}$  defined on some subspace  $W'$  of  $V$ , satisfying  $(\forall x \in W')(l(x) \leq p(x))$ , and for every  $z \in V \setminus W'$ , it is possible to extend  $l$  to a linear functional  $l' : W' + \mathbb{R}z \rightarrow \mathbb{R}$  such that for every  $x \in W' + \mathbb{R}z$ ,  $l'(x) \leq p(x)$ <sup>9</sup>. Thus, proceeding by induction, it can be shown that for every finite  $X \subseteq V$  there is an extension  $f'$  of  $f$ , defined in the subspace  $W'$  generated by  $W \cup X$ , such that  $(\forall x \in W')(f'(x) \leq p(x))$ . Consequently, by linearity of  $f'$  and positive homogeneity of  $p$ ,  $f'(x) \in [-p(-x), p(x)] = I(x)$ , so we can let  $g(x)$  be the leftmost dyadic subinterval of depth  $h(x)$  of  $I(x)$  containing  $f'(x)$ , and this way we have defined an element  $g \in A_h$ .  $\square$

We now define

$$\mathcal{A} = \{A_h \mid h : X \rightarrow \omega \text{ for some } X \in [V]^{<\omega}\},$$

and note that this is a centred family of finite antichains. For if we are given finitely many functions  $h_1 : X_1 \rightarrow \omega, \dots, h_n : X_n \rightarrow \omega$ , we can define  $h : (X_1 \cup \dots \cup X_n) \rightarrow \omega$  by  $h(x) = \max\{h_i(x) \mid 1 \leq i \leq n \text{ and } x \in X_i\}$ , and take a  $g \in A_h$  by Claim 3.8. Now for each  $1 \leq i \leq n$ , and each  $x \in X_i$ , we pick a dyadic interval  $g_i(x)$  of depth  $h_i(x)$  containing the interval  $g(x)$  (recall that  $g(x)$  is a dyadic interval of depth  $h(x) \geq h_i(x)$ ), so there is a unique such interval). Then it is clear that  $g$  extends the element  $g_i \in A_{h_i}$ .

Hence we can invoke an  $\mathcal{A}$ -generic 2-linked set  $G$ . We define  $\hat{f} : V \rightarrow \mathbb{R}$  as follows: for every  $x \in V$ , we let  $\mathcal{S}_x = \{g(x) \mid g \in G \text{ and } x \in \text{dom}(g)\} \subseteq I_\infty$ . For each  $n < \omega$ , since  $G$  must intersect  $A_{\{(x,n)\}}$ , it follows that  $\mathcal{S}_x$  contains at least one dyadic subinterval of  $I(x)$  of depth  $n$ ; and since  $G$  is 2-linked this interval is, in fact, unique. Now if  $n < m$  and  $J, J' \in \mathcal{S}_x$  are the two dyadic subintervals of depths  $n$  and  $m$ , respectively, then  $J' \subseteq J$ . Hence the family  $\mathcal{S}_x$  can be thought of as a nested sequence of closed intervals, with arbitrarily small diameters; therefore there is a unique real number  $\hat{f}(x) \in \bigcap \mathcal{S}_x$ . By construction,  $\hat{f}(x) \in I(x)$ , so  $\hat{f}(x) \leq p(x)$  for every  $x \in V$ . Also, given  $x \in W$ , we have that  $f(x) \in \bigcap \mathcal{S}_x$ , thus  $\hat{f}(x) = f(x)$ , so the function  $\hat{f}$  actually extends  $f$ . We now proceed to show that  $\hat{f}$  is a linear

<sup>9</sup>A proof of this fact can be found in [13, 2.3.2, p. 57].

functional, so let  $x, y \in V$  and suppose towards a contradiction that  $\hat{f}(x) + \hat{f}(y) \neq \hat{f}(x+y)$ . Then we can pick an  $n < \omega$  so large, that no interval of length  $\frac{p(x+y)+p(-x-y)}{2^n}$  containing  $\hat{f}(x+y)$  can intersect an interval of length  $\frac{p(x)+p(-x)+p(y)+p(-y)}{2^{n-1}}$  containing  $\hat{f}(x) + \hat{f}(y)$ . But then, letting  $h$  be the function constantly  $n$  with domain  $\{x, y, x+y\}$ , there must be a  $g \in G \cap A_h$ ; and so on the one hand we must have  $g(x) \in I(x)_n, g(y) \in I(y)_n, g(x+y) \in I(x+y)_n$  (so that  $g(x), g(y)$ , and  $g(x+y)$  are intervals of lengths  $\frac{p(x)+p(-x)}{2^n}, \frac{p(y)+p(-y)}{2^n}$ , and  $\frac{p(x+y)+p(-x-y)}{2^n}$ , respectively), and  $(g(x) + g(y)) \cap g(x+y) \neq \emptyset$ ; while simultaneously  $\hat{f}(x) + \hat{f}(y) \in g(x) + g(y)$  and  $\hat{f}(x+y) \in g(x+y)$ , which is a contradiction. Hence  $\hat{f}(x) + \hat{f}(y) = \hat{f}(x+y)$ ; and in a completely analogous way the reader can verify that  $\hat{f}(rx) = r\hat{f}(x)$  for every  $x \in V$  and every  $r \in \mathbb{R}$ . This finishes the proof.

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