Corrigendum

Corrigendum to "Gruff ultrafilters" [Topol. Appl. 210 (2016) 355–365]

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The authors regret that in Section 4 of the paper [2], the proof of Theorem 4.2 contains a mistake, and the authors do not know whether it can be fixed. The problem is that the terms A_α of the strong pathway $\langle A_\alpha \big| \alpha < \omega_1 \rangle$, as defined in the paper, will not be closed under binary set-theoretically definable operations; in fact, they will not even be closed under the join operation (the authors are very grateful to Osvaldo Guzmán for pointing this out, as well as for providing a concrete example of the failure of this closure property). In fact, the same problem arises in an analogous argument from the paper [1], where a similar technique was first implemented. Consequently, the question of whether there are P-points in every Random extension of a model of CH (an affirmative answer to which [1] had been considered an established result for quite some time) appears to be open at the moment.

Since the proof of Theorem 4.3 in [2] depends on Theorem 4.2, the authors regrettably withdraw the claim made in the statement of this Theorem. We do not know if there exist gruff ultrafilters in every model obtained by adding Random reals to a model of CH. We do, however, know that there is a model M of CH such that there are gruff ultrafilters in any model obtained by adding any number of random reals to M. To see this it suffices to consider $M = \mathbf{V}^{\mathcal{C}}$, where $\mathbf{V} \vDash \mathsf{CH}$ and $\mathcal{C}$ denotes the forcing notion for adding ω_1-many Cohen reals to $\mathbf{V}$. Let $\mathring{\mathcal{R}}$ be the ($\mathbf{V}^{\mathcal{C}}$-name for the) forcing notion that adds λ-many Random reals to $\mathbf{V}^{\mathcal{C}}$, where λ is some cardinal of $\mathbf{V}^{\mathcal{C}}$ (equivalently, of $\mathbf{V}$, since $\mathcal{C}$ is a c.c.c. forcing notion). Then we claim that in the generic extension $\mathbf{V}^{\mathcal{C}\star\mathring{\mathcal{R}}}$ there is a strong pathway.

To see this, let $\langle c_\alpha \big| \alpha < \omega_1 \rangle$ be the ω_1-sequence of Cohen reals added by $\mathcal{C}$. For each $\alpha \leq \omega_1$, consider the model of Set Theory $\mathbf{V}_\alpha = \mathbf{V}[\langle c_\xi \big| \xi < \alpha \rangle]$. If we let b^W denote the interpretation of the Borel code b in the model of Set Theory W, and we have two such models $W \subseteq V$, recall that a function $f \in 2^\lambda \cap V$ is *random* over W if $f \notin b^V$ for any null G_δ-set coded in W (i.e. such that $b \in W$). In particular, if G is an $\mathcal{R}$-generic filter over $\mathbf{V}_{\omega_1}$, and $r_G(\alpha) = i$ if and only if $[\{f \in 2^\lambda : f(\alpha) = i\}] \in G$, then $r_G \in \mathbf{V}_{\omega_1}[G]$ is random over $\mathbf{V}_{\omega_1}$, and consequently it is also random over $\mathbf{V}_\alpha$ for every $\alpha < \omega_1$. It is then easy to verify that the sequence $\langle A_\alpha \big| \alpha < \omega_1 \rangle$ given by

$$A_{\alpha+1} = \omega^\omega \cap \mathbf{V}_\alpha[r_G]$$

(for limit α, we let $A_\alpha = \bigcup_{\xi<\alpha} A_\xi$ in order to make the sequence continuous) will be a strong pathway: First, as $\mathbf{V}_\alpha[r_G]$ is a generic extension of $\mathbf{V}$, $A_{\alpha+1}$ has all the closure properties required; moreover, $c_\alpha \in \mathbf{V}_{\alpha+1} \setminus \mathbf{V}_\alpha$ is not dominated by any real from $\mathbf{V}_\alpha$ and hence it is not dominated by any element of $A_{\alpha+1}$ either. On the other hand, every real in $\mathbf{V}^{\mathcal{C}\star\mathring{\mathcal{R}}}$ only depends on r_G and countably many of the $\langle c_\alpha \big| \alpha < \omega_1 \rangle$, so

$$\omega^\omega = \bigcup_{\alpha<\omega_1} A_\alpha.$$

The proof of Theorem 4.3 goes through verbatim upon replacing the Random model with the model $\mathbf{V}_{\omega_1}[G]$, by using the strong pathway defined above. Hence we can conclude the existence of gruff ultrafilters in any model that arises from having CH, adding ω_1 Cohen reals, and subsequently adding some uncountable number of Random reals. (Incidentally, with the same reasoning we can conclude that there are P-points in any such model, which is an unpublished old result of Kunen).

The authors would like to apologise for any inconvenience caused.

References

[1] P. E. Cohen, *P-points in random universes.* Proc. Am. Math. Soc. **74** (2) (1979), 318–321.

[2] D. J. Fernández-Bretón and M. Hrušák, *Gruff ultrafilters.* Top. Appl. **210**(2016), 355–365.

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