

# Algebra and Topology in the Stone-Čech Compactification

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## 2 Introduction

The Stone-Čech compactification of discrete semigroups is a tool of central importance in several areas of mathematics, and has been studied extensively. We think of the Stone-Čech compactification of a discrete abelian semigroup  $G$  as the set  $\beta G$  of ultrafilters on  $G$ , where the point  $x \in G$  is identified with the principal ultrafilter  $\{A \subseteq G \mid x \in A\}$ , and the basic open sets are those of the form  $\bar{A} = \{p \in \beta G \mid A \in p\}$ , for  $A \subseteq G$ . Then these sets are actually clopen, and  $\bar{A}$  is really the closure in  $\beta G$  of the set  $A$ , regarded as a subset of  $\beta G$  under the aforementioned identification of points in  $G$  with principal ultrafilters. The semigroup operation  $+$  on  $G$  is also extended by the formula

$$p + q = \{A \subseteq G \mid \{x \in G \mid \{y \in G \mid x + y \in A\} \in q\} \in p\}$$

which turns  $\beta G$  into a right topological semigroup, meaning that for each  $p \in \beta G$  the mapping  $(\cdot) + p : \beta G \rightarrow \beta G$  is continuous (note that the extended operation  $+$  need not be commutative, and, even if  $G$  is a group, elements  $p \in G^* = \beta G \setminus G$  do not have inverses). The standard reference for this topic is [9].

The theory of the Stone-Čech compactification has helped simplify and sometimes even provide proofs of several combinatorial results that would enter the category of the so-called *Ramsey-type theorems*. Some of the most remarkable examples are:

**Theorem 1** (van der Waerden). *Suppose we partition the set of natural numbers  $\mathbb{N}$  into two cells. Then one of the cells must contain arbitrarily long arithmetic progressions.*

**Theorem 2** (Szemerédi). *Let  $A \subseteq \mathbb{N}$  be a set such that*

$$\limsup_{n \rightarrow \infty} \frac{|A \cap n|}{n} > 0$$

*( $A$  has positive upper density). Then  $A$  must contain arbitrarily long arithmetic progressions.*

Possibly the first striking application of these methods was the one that led to a very elegant proof of Hindman's finite sums theorem (formerly known as Graham-Rothschild's conjecture). For  $X \subseteq \mathbb{N}$ , define  $\text{FS}(X)$  to be the set of all sums of finitely many elements from  $X$ , where each element can be added only once:

$$\text{FS}(X) = \left\{ \sum_{x \in a} x \mid x \in [X]^{<\omega} \right\}.$$

**Theorem 3** (Hindman). *Let the set of natural numbers  $\mathbb{N}$  be partitioned into two cells. Then one of the cells must contain a set of the form  $\text{FS}(X)$  for some infinite  $X \subseteq \mathbb{N}$ .*

There are two key points that allow us to prove this theorem. The first one is that the semigroup compactification  $\beta\mathbb{N}$  must have idempotent elements. This is a consequence of the so-called *Ellis-Numakura lemma*: every compact right-topological semigroup has idempotents. The second point is that if the ultrafilter  $p \in \mathbb{N}^*$  is idempotent, then every  $A \in p$  must contain a set of the form  $\text{FS}(X)$  for some infinite subset  $X \subseteq \mathbb{N}$ . When Hindman was trying to prove this result, he was very much aware of this fact. However, he still did not know that the existence of idempotent ultrafilters can be proved in ZFC (the usual axioms of mathematics), but he was able to construct them by adding the additional assumption that CH (the Continuum Hypothesis) holds ([6, Th. 3.3]). Later on, he proved his theorem (unconditionally) using combinatorial methods, and afterwards, the Ellis-Numakura result started to be well-known, so it seemed as though Hindman's CH construction should face oblivion. However, van Douwen pointed out that not only were the ultrafilters constructed by Hindman idempotent, but they in fact have a stronger property, captured by the following definition.

**Definition 4.** *If  $p \in G^*$ , we say that  $p$  is **strongly summable** if it has a basis of sets of the form  $\text{FS}(X)$ , i.e. if for every  $A \in p$  there exists an infinite  $X \subseteq G$  such that  $p \ni \text{FS}(X) \subseteq A$ .*

When the semigroup under consideration is  $\mathbb{N}$ , then strongly summable ultrafilters are always idempotent. They also have several nice algebraic properties, notably amongst them the following.

**Theorem 5.** *Let  $p \in \mathbb{N}^*$  be a strongly summable ultrafilter. Then  $p$  has the **weak trivial sums property**, i.e. whenever  $q, r \in \mathbb{N}^*$  are such that  $q + r = r + q = p$ , then  $q, r \in \mathbb{Z} + p$ .*

Thus in particular, the biggest subgroup of  $\mathbb{N}^*$  which has  $p$  as the identity element (this can be defined for any idempotent on a semigroup) is just a copy of  $\mathbb{Z}$ .

In [7], the authors generalize some results previously only known to hold for ultrafilters on  $\beta\mathbb{N}$  or  $\beta\mathbb{Z}$ . In particular, they proved there that every strongly summable ultrafilter  $p$  on any abelian group  $G$  is an idempotent ([7, Th. 2.3]). And [7, Th. 4.6] states that if  $G$  can be embedded in the group  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  and  $p \in G^*$  is strongly summable, then it has the weak trivial sums property.

In order to get stronger properties, it is possible to strengthen the definition of strongly summable.

**Definition 6.** *An ultrafilter  $p \in \beta G$  is **sparse** if for every  $A \in p$  there are two sets  $Y \subseteq X$  such that  $X \setminus Y$  is infinite,  $\text{FS}(X) \subseteq A$ , and  $\text{FS}(Y) \in p$ .*

**Theorem 7** ([7, Th. 4.5]). *If  $G$  can be embedded in  $\mathbb{T}$  and  $p \in G^*$  is sparse, then  $p$  has the **trivial sums property**, i.e. whenever  $q, r \in G^*$  are such that  $q + r = p$ , it must be the case that  $q, r \in G + p$ .*

In [8], the authors investigate the different kinds of abelian semigroups on which every nonprincipal strongly summable ultrafilter must be sparse. In particular, [8, Th. 4.2] establishes that if  $S$  is a

subsemigroup of  $\mathbb{T}$ , then every nonprincipal strongly summable ultrafilter on  $S$  is sparse. After, they built on this to prove a more general result. We identify elements of  $\mathbb{T}$  (which are cosets of real numbers modulo  $\mathbb{Z}$ ) with their unique representative  $t$  satisfying  $0 \leq t < 1$ .

**Theorem 8** ([8], Th. 4.5 and Cor. 4.6). *Let  $S$  be a countable subsemigroup of  $\bigoplus_{n < \omega} \mathbb{T}$  and let  $p$  be a nonprincipal strongly summable ultrafilter on  $S$ . If*

$$\left\{ x \in S \mid \pi_{\min(x)}(x) \neq \frac{1}{2} \right\} \in p$$

(here  $\min(x)$  denotes the least  $i$  such that  $\pi_i(x)$  is nonzero), or

$$\{x \in S \mid (\exists n \in \mathbb{N})(o(x) = 2^n)\} \notin p$$

(here  $o(x)$  denotes the order of  $x$ , i.e. the least  $n$  such that  $nx = 0$ ), then  $p$  is sparse.

Although it is not immediately clear that, for groups that are not embeddable in  $\mathbb{T}$ , sparseness implies the trivial sums property, Hindman, Steprāns and Strauss were able to get an analogous result concerning the latter property.

**Theorem 9** ([8], Th. 4.8 and Cor. 4.9). *Let  $S$  be a countable subsemigroup of  $\bigoplus_{n < \omega} \mathbb{T}$  and let  $p$  be a nonprincipal strongly summable ultrafilter on  $S$ . If*

$$\left\{ x \in S \mid \pi_{\min(x)}(x) \neq \frac{1}{2} \right\} \in p,$$

or

$$\{x \in S \mid (\exists n \in \mathbb{N})(o(x) = 2^n)\} \notin p,$$

then  $p$  has the trivial sums property.

This was the state of affairs when I started working on these and related questions.

### 3 Results Achieved

Since the hypothesis of Theorems 8 and 9 involve a condition having to do with  $p$  concentrating on the set of elements having certain coordinates not equal to  $\frac{1}{2}$ , it was natural to ask what happens in the exact opposite situation, namely when  $p$  concentrates on the set of elements all of whose nonzero coordinates equal  $\frac{1}{2}$ . This means that  $p$  contains a copy of the (unique up to isomorphism) countable Boolean group. Thus Hindman, Steprāns and Strauss asked (in the originally submitted version of [8], though in the final published version they removed the question because of the following theorem) whether every strongly summable ultrafilter on the Boolean group is sparse. I was able to prove the following theorem, which answers that question.

**Theorem 10** ([3], Th. 2.1). *If  $G$  is the Boolean group, and  $p \in G^*$  is a strongly summable ultrafilter, then  $p$  is sparse.*

The analogous question concerning the trivial sums property had been answered long time ago by Protasov, who proved ([11, Cor. 4.4]) a more general result that implies that strongly summable ultrafilters on the Boolean group always have the trivial sums property. Thus, what remained open

was the corresponding general questions about whether or not every strongly summable ultrafilter on an arbitrary abelian group should be sparse and/or have the trivial sums property. These questions were explicitly asked in the paper by Hindman, Steprāns and Strauss ([8, Questions 4.11 and 4.12]), and the following theorem of mine answers them both.

**Theorem 11** ([4], Cor. 3.10 and 3.11). *Let  $G$  be any abelian group, and let  $p \in G^*$  be a strongly summable ultrafilter. Then  $p$  is sparse and has the trivial sums property.*

Looking into the finer structure of the proofs of some of the preceding theorems, the notion of a union ultrafilter and of an additive isomorphism emerged, and their importance became apparent. This notions were first introduced by Blass in [1].

**Definition 12.**

- For  $X$  a pairwise disjoint family of finite subsets of  $\omega$ , we define the set of finite unions of  $X$  as

$$\text{FU}(X) = \left\{ \bigcup_{x \in a} x \mid x \in [X]^{<\omega} \right\}$$

- A nonprincipal ultrafilter  $p$  on the set  $[\omega]^{<\omega}$  is a **union ultrafilter** if it has a basis of sets of the form  $\text{FU}(X)$ , i.e. if for every  $A \in p$  there exists a pairwise disjoint family  $X$  such that  $p \ni \text{FU}(X) \subseteq A$ .
- If  $G$  is an abelian group and  $p \in G^*$  is a strongly summable ultrafilter, and if  $q$  is a union ultrafilter, then we will say that  $p$  is **additively isomorphic** to  $q$  if there is a set  $X \subseteq G$  satisfying uniqueness of finite sums (i.e.  $\sum_{x \in a} x = \sum_{x \in b} x \Rightarrow a = b$  for  $a, b \in [X]^{<\omega}$ ) and a pairwise disjoint family  $Y$  such that  $\text{FS}(X) \in p$ ,  $\text{FU}(Y) \in q$  and there is a bijection  $f : X \rightarrow Y$  such that the mapping  $\varphi : \text{FS}(X) \rightarrow \text{FU}(Y)$  given by  $\sum_{x \in a} x \mapsto \bigcup_{y \in f[a]} y$  sends  $p$  to  $q$ .

As it turns out, the proof of Theorem 11 yields also the following result.

**Theorem 13** ([4], Cor. 3.2). *Let  $G$  be any abelian group, and let  $p \in G^*$  be a strongly summable ultrafilter. If  $\{x \in G \mid o(x) = 2\} \notin p$ , then  $p$  is additively isomorphic to a union ultrafilter.*

The argument for proving Theorem 13 easily breaks down when considering ultrafilters on the Boolean group, so again, it was natural to ask what happens in that case. Is every strongly summable ultrafilter on the Boolean group additively isomorphic to some union ultrafilter? The answer turned out to be negative.

**Theorem 14** ([4], Th. 4.12). *Assuming  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ , there exists a strongly summable ultrafilter on the Boolean group that is not additively isomorphic to any union ultrafilter.*

### 3.1 The nonabelian case

In joint work with Martino Lupini, an investigation of how much of the theory of strongly summable ultrafilters can be carried out in some restricted class of nonabelian groups is underway. So far, the most significant result is the following.

**Theorem 15** ([5]). *If  $G$  is a solvable group, and  $p \in G^*$  is a strongly summable ultrafilter, then  $p$  is idempotent.*

## 4 Further Questions

There are several questions that remain open and are related to the previous results. The main objective of my Ph. D. dissertation is to answer a reasonable subset of these questions. The very broad question about how much of this theory can be carried out in a nonabelian context would be the first one worth mentioning. I intend to keep investigating this question. Another question that I am trying to answer is a consistency result. It is known that the existence of strongly summable ultrafilters cannot be proved in ZFC, but it can be proved if we assume additional hypotheses such as the Continuum Hypothesis CH, Martin's Axiom MA and even a weakening of Martin's Axiom that is usually denoted by  $\text{cov}(\mathcal{M}) = \mathfrak{c}$  (Martin's axiom for countable forcing notions). I am currently trying to construct a model of ZFC where the hypothesis  $\text{cov}(\mathcal{M}) = \mathfrak{c}$  fails, yet there exist strongly summable ultrafilters. If successful, such a construction would be significant.

The other questions that I am trying to answer can be broadly arranged in two categories.

### 4.1 About union ultrafilters

An **ordered union** ultrafilter is an ultrafilter  $p$  on the set  $[\omega]^{<\omega}$  such that for every  $A \in p$  there exists an **ordered family**  $X$  (i.e.  $X$  is a pairwise disjoint family and for any two  $x, y \in X$  either  $\max(x) < \min(y)$  or  $\max(y) < \min(x)$ ) such that  $p \ni \text{FU}(X) \subseteq A$ . Ordered union ultrafilters were first introduced in [1], and there are at least two constructions ([2, Th. 4] and [10, Cor. 5.2] of union ultrafilters that are not ordered union (in fact, not additively isomorphic to any ordered union ultrafilter). If we define the **core** of an ultrafilter on  $[\omega]^{<\omega}$  to be the filter (which will never be maximal)  $\{\bigcup A \mid A \in p\}$ , then it is easy to prove that the core of an ordered union ultrafilter must be nonmeagre. But this is also true of the unordered union ultrafilters that we just mentioned. Thus we conjecture that the orderedness condition can be removed from the hypothesis of that result, namely we intend to prove that the core of every union ultrafilter must be nonmeagre. Another question is whether every union ultrafilter must be stable (stability is a condition whose definition is analogous to that of a P-point for the case of union ultrafilters, and every construction known so far of a union ultrafilter actually yields a stable union ultrafilter). The conjecture here is that it should be possible to construct, assuming a reasonable hypothesis such as  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ , a nonstable union ultrafilter. Whether such an ultrafilter could be ordered would be another question worth pursuing. I am investigating these questions jointly with David Chodounský.

### 4.2 Points of continuity of the sum in $\beta G$

The extension of the group operation of  $G$  to  $\beta G$  is such that for every ultrafilter  $q$ , the mapping  $p \mapsto p + q$  is continuous, however the mapping  $p \mapsto q + p$  is not. But P-points are always points of continuity of the latter map, for every  $q$ . Protasov [12, Problem 2] asked whether there are ultrafilters  $p, q$  such that the mapping  $r \mapsto q + r$  is continuous at  $p$ , yet  $p$  is not a P-point. Juris Steprāns conjectures that the answer to this question is consistently affirmative. Moreover he conjectures that it should be possible to construct a model of ZFC without P-points but where there exist points of continuity of the mapping  $r \mapsto q + r$  for some  $q$ . I am going to investigate these and other related questions as well.

## References

- [1] Blass, A., *Ultrafilters Related to Hindman's Finite Unions Theorem and its Extensions*, in Simpson, S. (ed.), *Logic and Combinatorics*; Contemporary Math. **65** (1987), 89-124.
- [2] Blass, A.; Hindman, N. *On Strongly Summable Ultrafilters and Union Ultrafilters*, Trans. Amer. Math. Soc. **304** No. 1 (1987), 83-99. MR0906807 (88i:03080), Zbl 0643.03032
- [3] Fernández Bretón, D. *Every Strongly Summable Ultrafilter on  $\bigoplus \mathbb{Z}_2$  is Sparse*, New York J. Math. **19** (2013), 117-129.
- [4] Fernández Bretón, D. *Strongly Summable Ultrafilters, Union Ultrafilters, and the Trivial Sums Property*, preprint. Available on the ArXiv at <http://arxiv.org/abs/1306.5421>, or from <http://math.yorku.ca/~davidfb/>.
- [5] Fernández Bretón, D; Lupini, M. *Strongly Summable Ultrafilters on Solvable Groups*, in preparation.
- [6] Hindman, N. *The existence of certain ultrafilters on  $\mathbb{N}$  and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc. **36** (1972), 341-346. MR0307926 (46 #7041), Zbl 0259.10046.
- [7] Hindman, N.; Protasov, I.; Strauss, D. *Strongly Summable Ultrafilters on Abelian Groups*, Matematychni Studii **10** (1998), 121-132. MR1687143 (2001d:22003), Zbl 0934.22005.
- [8] Hindman, N.; Steprāns, J.; Strauss, D. *Semigroups in which all Strongly Summable Ultrafilters are Sparse*, New York J. Math. **18** (2012), 835-848. Zbl pre06098874.
- [9] Hindman, N.; Strauss, D. *Algebra in the Stone-Čech Compactification*. de Gruyter Expositions in Mathematics 27. Walter de Gruyter, Berlin-New York, 1998. MR1642231 (99j:54001), Zbl 0918.22001.
- [10] Krautzberger, P. *On Union Ultrafilters*, Order **29** (2012), 317-343.
- [11] Protasov, I. *Finite groups in  $\beta G$* , Matematychni Studii **10**, (1998), 17-22.
- [12] Protasov, I. *The Topological Center of the Semigroup of Free Ultrafilters*, Mathematical Notes **63** (1998), 384-387.