

# GRUFF ULTRAFILTERS

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ABSTRACT. We investigate the question of whether  $\mathbb{Q}$  carries an ultrafilter generated by perfect sets (such ultrafilters were called *gruff ultrafilters* by van Douwen). We prove that one can (consistently) obtain an affirmative answer to this question in three different ways: by assuming a certain parametrized diamond principle, from the cardinal invariant equality  $\mathfrak{d} = \mathfrak{c}$ , and in the Random real model.

## 1. INTRODUCTION

In a 1992 paper, Eric van Douwen [7] carried out an investigation about certain points in the Čech-Stone compactification of  $\mathbb{Q}$  (where  $\mathbb{Q}$  is equipped with the topology inherited from the Euclidean topology on  $\mathbb{R}$ , so that points in  $\beta\mathbb{Q}$  can be realized as maximal filters of closed sets), with the property that they actually generate an ultrafilter on  $\mathbb{Q}$ . In other words, van Douwen was looking at ultrafilters over  $\mathbb{Q}$  that have a base of closed sets, and among those he paid particular attention to the ones where the elements of a base can be taken to be crowded (recall that a set is crowded if it has no isolated points), in addition to being closed. This was the motivation for stating the following definition.

**Definition 1.1.** A nonprincipal ultrafilter  $u$  on  $\mathbb{Q}$  is said to be **gruff** (a pun on the fact that these are points in  $\beta\mathbb{Q}$  that “generate real ultra filters”) if it has a base of perfect (i.e. closed and crowded) subsets of  $\mathbb{Q}$ . This is, we require that  $(\forall A \in u)(\exists X \in u)(X \text{ is perfect and } X \subseteq A)$ .

Recall that a coideal on a set  $X$  is a family  $\mathcal{A}$  with the property that  $\emptyset \notin \mathcal{A}$ ,  $\mathcal{A}$  is closed under supersets, and whenever an element  $A \in \mathcal{A}$  is written as  $A = A_0 \cup A_1$ , there exists an  $i \in 2$  such that  $A_i \in \mathcal{A}$ . If the infinite set  $X$  has a topology in which  $X$  itself is crowded, then the family

$$\mathcal{C} = \{A \subseteq X \mid A \text{ contains an infinite crowded set}\}$$

constitutes a coideal. Moreover, in the topological space  $\mathbb{Q}$ , every infinite crowded set contains an infinite perfect subset. This fact, which is not true in a general topological space (for example, in every Polish space it is possible to construct *Bernstein sets*, sets that are not contained in nor disjoint from any uncountable perfect subset), implies that the family

$$\mathcal{P} = \{A \subseteq X \mid A \text{ contains an infinite perfect set}\},$$

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2010 *Mathematics Subject Classification.* Primary 03E75; Secondary 54A35, 03E35, 03E17, 54F05.

*Key words and phrases.* Ultrafilter, Random forcing, dominating number, parametrized diamond principle, topology on the rationals.

The first author was partially supported by Postdoctoral Fellowship number 263820 from the Consejo Nacional de Ciencia y Tecnología (CONACyT), Mexico; while the research of the second author was partially supported by PAPIIT grant IN 108014 and CONACyT grant 177758.

also constitutes a coideal on  $\mathbb{Q}$ . It is for this reason that Definition 1.1 is justified.

The main question that van Douwen asked about gruff ultrafilters is whether their existence can be proved in ZFC. He himself [7, Thm. 2.1] provided a partial answer by proving that the existence of a gruff ultrafilter follows from the cardinal invariant equality  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ , which is equivalent to Martin's Axiom restricted to countable forcing notions. Although the question remains open, more partial results have been proven. Copláková and Hart [6, Thm. 1] proved in 1999 that the existence of a gruff ultrafilter follows from  $\mathfrak{b} = \mathfrak{c}$ . Some time after, in 2003, Ciesielski and Pawlikowski [3, Thm. 4.22] showed that the existence of a gruff ultrafilter follows from a combinatorial principle known as  $\text{CPA}_{\text{prism}}^{\text{game}}$ , which, in particular, implies that there exist gruff ultrafilters in the Sacks model, as this model satisfies that combinatorial principle. This theorem was improved shortly after by Millán [8, Thm. 3], who showed that, in fact,  $\text{CPA}_{\text{prism}}^{\text{game}}$  implies the existence of a gruff ultrafilter that is at the same time a  $\mathbb{Q}$ -point (it is shown in [4, Prop. 5.5.5] that a gruff ultrafilter cannot be a P-point).

In this paper, we obtain three more partial answers to van Douwen's question. The first result involves the theory of diamond principles that are parametrized by a cardinal invariant, as developed in [9]. We define a cardinal characteristic  $\mathfrak{r}_P$  that relates naturally to perfect subsets of  $\mathbb{Q}$ , and show that its corresponding parametrized diamond principle  $\diamond(\mathfrak{r}_P)$  implies the existence of a gruff ultrafilter which is at the same time a  $\mathbb{Q}$ -point. We also show that this parametrized diamond principle holds in the Sacks model, thus providing an alternative proof of Millán's theorem on the existence of gruff  $\mathbb{Q}$ -points in this model. Our second result is that the existence of gruff ultrafilters follows from the cardinal invariant equality  $\mathfrak{d} = \mathfrak{c}$ . Since (it is provable in ZFC that)  $\mathfrak{d} \geq \mathfrak{b}$  and  $\mathfrak{d} \geq \text{cov}(\mathcal{M})$ , but both inequalities can be consistently strict (even simultaneously), this result is stronger than both van Douwen's, and Copláková and Hart's. Finally, our third result is that in the Random real model there exists a gruff ultrafilter (this, together with the  $\diamond(\mathfrak{r}_P)$  result mentioned above, shows that the existence of gruff ultrafilters is consistent with  $\mathfrak{d} < \mathfrak{c}$ ). First we will prove a lemma that will simplify our interaction with gruff ultrafilters. Throughout this paper, the notation  $(a, b)$  will be used to refer to intervals on  $\mathbb{Q}$ . In other words,  $(a, b) = \{q \in \mathbb{Q} \mid a < q < b\}$  whenever  $a, b \in \mathbb{R}$ .

**Lemma 1.2.** *There exists a gruff ultrafilter on  $\mathbb{Q}$  if and only if there exists an ultrafilter on the set of positive rational numbers  $\mathbb{Q}^+$  with a base of perfect unbounded sets.*

*Proof.* The “if” part is obvious. For the other direction, assume that  $u$  is a gruff ultrafilter on  $\mathbb{Q}$ . Without loss of generality,  $u$  concentrates on  $\mathbb{Q}^+$ , since otherwise it would have to concentrate on the set of negative rational numbers  $\mathbb{Q}^-$ , in which case the function  $x \mapsto -x$  (which is an autohomeomorphism of  $\mathbb{Q}$ ) will map  $u$  to another gruff ultrafilter  $-u$  that concentrates on  $\mathbb{Q}^+$ .

Hence we get to assume that  $u$  is an ultrafilter on  $\mathbb{Q}^+$  with a base of perfect sets. If all of these perfect sets happen to be unbounded, we are done. Otherwise  $u$  contains a bounded set, and hence  $u$  must converge to some real number  $r$ . If  $u$  concentrates on  $(0, r)$ , then take an order-isomorphism  $f : (0, r) \rightarrow \mathbb{Q}^+$ , and notice that  $f$  will be a homeomorphism mapping every element of  $u$  to an unbounded set. Otherwise,  $u$  must concentrate on  $(r, \infty)$ , and so we can take an order anti-isomorphism  $f : (r, \infty) \rightarrow \mathbb{Q}^+$ , which will map every element of  $u$  to

an unbounded set. In either case, the ultrafilter  $f(u)$  will have a base of perfect unbounded sets.  $\square$

Throughout the rest of this paper, we will write  $\mathbb{Q}$  instead of  $\mathbb{Q}^+$ . Similarly,  $\mathcal{P}$  will now denote the coideal of subsets of  $\mathbb{Q}$  (i.e.  $\mathbb{Q}^+$ ) containing an *unbounded* perfect set, and we will take gruff to mean an ultrafilter on  $\mathbb{Q}$  (i.e.  $\mathbb{Q}^+$ ) with a base of sets that are both perfect and unbounded. Lemma 1.2 guarantees that these changes do not make any difference regarding the question of the existence of gruff ultrafilters.

## 2. A CARDINAL INVARIANT AND ITS PARAMETRIZED DIAMOND PRINCIPLE

We will start by laying down some terminology and results from [9] that we will be using throughout this section. The theory of parametrized diamond principles involves cardinal invariants given by triples  $(A, B, E)$  where  $|A| \leq \mathfrak{c}$ ,  $|B| \leq \mathfrak{c}$  and  $E \subseteq A \times B$ , where we additionally require that  $(\forall a \in A)(\exists b \in B)(a E b)$  and that  $(\forall b \in B)(\exists a \in A)\neg(a E b)$  (these last two requirements are only to ensure existence and nontriviality of the corresponding cardinal invariant). The cardinal invariant associated to such a triple  $(A, B, E)$  (sometimes referred to as the *evaluation* of the triple), denoted by  $\langle A, B, E \rangle$ , is given by

$$\langle A, B, E \rangle = \min\{|X| : X \subseteq B \wedge (\forall a \in A)(\exists b \in X)(a E b)\}.$$

We consider cardinal invariants  $\langle A, B, E \rangle$  that are Borel, which we take to mean that all three entries of the corresponding triple  $(A, B, E)$ , can be viewed as Borel subsets of some Polish space. Then the *parametrized diamond principle* of the triple is the statement that for every Borel function  $F : 2^{<\omega_1} \rightarrow A$  (where Borel means that for every  $\alpha < \omega_1$  the restriction  $F \upharpoonright 2^{<\alpha}$  is a Borel function) there exists a  $g : \omega_1 \rightarrow B$  such that for every  $f : \omega_1 \rightarrow 2$ , it is the case that  $F(f \upharpoonright \alpha) E g(\alpha)$  for stationarily many  $\alpha < \omega_1$ . This statement is denoted by  $\diamond(A, B, E)$ . The fundamental theorem regarding these parametrized diamond principles is the following

**Theorem 2.1** ([9], Thm. 6.6). *Let  $(A, B, E)$  be a Borel triple defining a cardinal invariant. Let  $\mathbb{P}$  be a Borel forcing notion such that  $\mathbb{P} \simeq 2^+ \times \mathbb{P}$ , and let  $\langle \mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$  be a countable support iteration such that for every  $\alpha$ ,  $\mathbb{P}_\alpha \Vdash \text{“}\mathring{\mathbb{Q}}_\alpha = \mathbb{P}\text{”}$ . Assume also that the final step of the iteration,  $\mathbb{P}_{\omega_2}$ , is proper. Then*

$$\mathbb{P}_{\omega_2} \Vdash \text{“}\diamond(A, B, E)\text{” if and only if } \mathbb{P}_{\omega_2} \Vdash \text{“}\langle A, B, E \rangle \leq \omega_1\text{”}.$$

We will say that a set  $X$  is *two-sided crowded* if for every  $x \in X$  and every  $\varepsilon > 0$  there are points  $y, z \in X$  such that  $x - \varepsilon < y < x < z < x + \varepsilon$ . A two-sided perfect set would be a closed, two-sided crowded set. In what follows, we will say that  $X \subseteq \mathbb{Q}$  is scattered if no subset of  $X$  is two-sided crowded. We let

$$\mathcal{B} = \{X \subseteq \mathbb{Q} \mid X \text{ is two-sided perfect and unbounded}\},$$

and note that  $\mathcal{B}$  generates a coideal  $\mathcal{P}' \subseteq \mathcal{P}$ .

We consider here the cardinal invariant  $\mathfrak{r}_P$  to be the evaluation of the triple  $(\mathcal{P}(\mathbb{Q}), \mathcal{B}, R)$ , where the relation “ $Y R X$ ” ( $X$  *reaps*  $Y$  modulo  $\mathcal{P}'$ , or  $Y$  is *reaped* by  $X$  modulo  $\mathcal{P}'$ ) means that either  $X \setminus Y \notin \mathcal{P}'$  or  $X \cap Y \notin \mathcal{P}'$  ( $X$  is either contained in, or disjoint from,  $Y$ , modulo  $\mathcal{P}'$ ). In other words,  $\mathfrak{r}_P = \mathfrak{r}(\mathcal{P}(\mathbb{Q}) \times \omega) / (\text{scattered} \times \text{fin})$ . Equivalently,  $\mathfrak{r}_P$  is the least cardinality of a family  $\mathcal{X}$  of unbounded two-sided perfect subsets of  $\mathbb{Q}$  such that for every colouring of the

elements of  $\mathbb{Q}$  into two colours, there exists an element of  $\mathcal{X}$  which is monochromatic, except possibly for a scattered or a bounded subset.

Hence the combinatorial principle  $\diamond(\tau_P)$  is the statement that for every Borel function  $F : 2^{<\omega_1} \rightarrow \mathcal{P}(\mathbb{Q})$ , there exists a function  $g : \omega_1 \rightarrow \mathcal{B}$  (this is, an  $\omega_1$ -sequence of two-sided perfect unbounded subsets of  $\mathbb{Q}$ ) satisfying that for every  $f : \omega_1 \rightarrow 2$ ,  $g(\alpha)$  will reap  $F(f \upharpoonright \alpha)$  modulo  $\mathcal{P}'$  for stationarily many  $\alpha < \omega_1$ .

In order to use the combinatorial principle  $\diamond(\tau_P)$ , we need a definition and a lemma.

**Definition 2.2.** If  $\mu$  is an ordinal, we say that a  $\mu$ -sequence  $\langle X_\alpha \mid \alpha < \mu \rangle$  of subsets of  $\mathbb{Q}$  is *descending modulo  $\mathcal{P}'$*  if every  $X_\alpha \in \mathcal{P}'$  and, whenever  $\xi < \alpha < \mu$ , we have that  $X_\alpha \setminus X_\xi \notin \mathcal{P}'$ .

**Lemma 2.3.** *Let  $\langle X_n \mid n < \omega \rangle$  be a descending  $\omega$ -sequence modulo  $\mathcal{P}$ . Then it is possible to choose, in a Borel way, a two-sided perfect set  $X \in \mathcal{P}$  that is almost contained in every  $X_n$  modulo  $\mathcal{P}'$  (this is,  $X \setminus X_n \notin \mathcal{P}'$ ).*

*Proof.* First define, for  $n < \omega$ , the interval  $I_n = (n\sqrt{2}, (n+1)\sqrt{2})$ . Then each of the  $I_n$  will be a clopen interval in  $\mathbb{Q}$ , and  $\mathbb{Q} = \bigcup_{n < \omega} I_n$ . Note that, for every  $n < \omega$ , since  $X_n \in \mathcal{P}'$  then we must have that  $X_n \cap I_n$  is not scattered for infinitely many  $n < \omega$ . Similarly, for  $n < m < \omega$ , we have that  $(X_m \setminus X_n) \cap I_k$  must be scattered for almost all  $k$ . Hence we recursively construct an increasing sequence  $\langle k_n \mid n < \omega \rangle$ , and non-scattered sets  $B_n \subseteq I_{k_n}$ , as follows:  $k_0$  is any number such that  $I_{k_0} \cap X_0$  is not scattered, and we let  $B_0 = I_{k_0} \cap X_0$ . Suppose we have picked  $k_0, \dots, k_n$  and  $B_0, \dots, B_n$  such that  $B_j \setminus X_i$  is scattered whenever  $i \leq j \leq n$ . Then we pick a  $k_{n+1} > k_n$  such that  $(X_{n+1} \setminus X_i) \cap I_{k_{n+1}}$  is scattered whenever  $i \leq n$ , and we let  $B_{n+1} = X_{n+1} \cap I_{k_{n+1}}$ .

Now we choose, in a Borel way, a two-sided perfect subset  $P_n \subseteq B_n$ . To do this, we first fix  $C_n$  to be the maximal two-sided crowded subset of  $B_n$  (which is the union of all two-sided crowded subsets of  $B_n$ ). Now we fix an effective enumeration  $\{q_k \mid k < \omega\}$  of  $\mathbb{Q}$ , and recursively define finite sets  $F_k \subseteq B_n$  (for all  $k < \omega$ ) and clopen intervals  $I_k$  (for  $q_k \notin C_n$ ), as follows: at stage  $k$ , we first choose, for each  $x \in F_{k-1}$ , the least-indexed  $y, z \in C_n$  which are within  $\frac{1}{2^k}$  of  $x$ , with  $y < x < z$ , and which do not belong to any  $I_i$  for  $i < k$ , and put all those  $x, y, z$ , into  $F_k$ . Afterwards, if  $q_k \notin C_n$  then we let  $I_k$  be a clopen interval centred around  $q_k$  which does not intersect  $F_k$  (and otherwise we do not define  $I_k$ ). This way, in the end we get the two-sided perfect set  $P_n = \bigcup_{k < \omega} F_k \subseteq C_n$ , which is two-sided crowded by construction, and closed because its complement is exactly  $\bigcup_{\substack{k < \omega \\ q_k \notin C_n}} I_k$ .

In the end, we define  $X = \bigcup_{n < \omega} P_n$ , and we are done.  $\square$

**Theorem 2.4.**  $\diamond(\tau_P)$  implies the existence of a gruff ultrafilter.

*Proof.* By suitable coding, we consider elements of  $2^{<\omega_1}$  that represent pairs  $\langle \vec{A}, A \rangle$  such that  $A \subseteq \mathbb{Q}$  and  $\vec{A} = \langle A_\xi \mid \xi < \alpha \rangle$  is a sequence of two-sided perfect unbounded subsets of  $\mathbb{Q}$  that is descending modulo  $\mathcal{P}$ . We choose an increasing sequence  $\langle \alpha_n \mid n < \omega \rangle$ , cofinal in  $\alpha$ , and we build, in a Borel way, a two-sided perfect unbounded set  $B \subseteq \mathbb{Q}$  satisfying that  $(\forall n < \omega)(B \setminus A_{\alpha_n} \notin \mathcal{P})$ , using Lemma 2.3. Since  $B$  is two-sided crowded, it is a countable linear order with no endpoints, which means that we can map  $B$  homeomorphically to  $\mathbb{Q}$  in a Borel way (by constructing an order-isomorphism between  $B$  and  $\mathbb{Q}$  in the usual back-and-forth way)

and define  $F(\langle \vec{A}, A \rangle)$  to be either the image of  $B \cap A$  under this mapping, if that image belongs to  $\mathcal{P}'$ , or the image of  $B \setminus A$  otherwise (in this construction, if  $\alpha$  is a successor cardinal,  $\alpha = \xi + 1$ , then there is no need to pick a cofinal sequence, and we can let  $B = A_\xi$  and perform the rest of the construction in the exact same way). Use  $\diamond(\mathfrak{r}_P)$  to get a  $g : \omega_1 \rightarrow \mathcal{B}$  satisfying that for every  $f : \omega_1 \rightarrow 2$ ,  $F(f \upharpoonright \alpha)$  is reaped, modulo  $\mathcal{P}'$ , by the perfect unbounded set  $g(\alpha)$  for stationarily many  $\alpha < \omega_1$ . We use  $g$  to recursively construct our gruff ultrafilter.

So assume that we have constructed a sequence of two-sided perfect unbounded sets  $\langle X_\xi \mid \xi < \alpha \rangle$  that is descending modulo  $\mathcal{P}'$ . Using the same cofinal sequence  $\langle \alpha_n \mid n < \omega \rangle$  as in the previous paragraph, and in the exact same Borel way, construct a two-sided perfect  $X$  which is contained, modulo  $\mathcal{P}'$ , in each  $X_{\alpha_n}$ , and map it homeomorphically onto  $\mathbb{Q}$ . We let  $X_\alpha \subseteq X$  be the preimage of  $g(\alpha)$  under that homeomorphism, so that  $X_\alpha$  is a perfect unbounded subset of  $\mathbb{Q}$  (if  $\alpha = \xi + 1$  then we let  $X = X_\xi$  and perform the rest of the construction in the exact same way). This gives us an  $\omega_1$ -sequence of perfect unbounded sets, descending modulo  $\mathcal{P}'$ ,  $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ . Clearly the members of this sequence generate a filter, which then will be gruff provided this filter is an ultrafilter. To see that the filter generated by  $\{X_\alpha \mid \alpha < \omega_1\}$  is indeed an ultrafilter, let  $A \subseteq \mathbb{Q}$  and let  $f : \omega_1 \rightarrow 2$  be the branch of  $2^{\omega_1}$  that represents  $(\vec{X}, A)$  under the relevant coding. Then by choice of  $g$ , for stationarily many  $\alpha < \omega_1$  we will have that  $F(\vec{X} \upharpoonright \alpha, A)$  is reaped modulo  $\mathcal{P}$  by  $g(\alpha)$ . But  $F(\vec{X} \upharpoonright \alpha, A)$  is the image of either  $X \cap A$  or of  $X \setminus A$  under the Borel homeomorphism onto  $\mathbb{Q}$  that was obtained at stage  $\alpha$ , where  $X$  is the set obtained at that stage using Lemma 2.3; and  $X_\alpha$  is the preimage of  $g(\alpha)$  under that same homeomorphism. Hence we can conclude that  $X_\alpha$  reaps  $A$  modulo  $\mathcal{P}'$ . Since the filter generated by  $\{X_\alpha \mid \alpha < \omega_1\}$  consists only of elements in  $\mathcal{P}'$ , then we can conclude that such a filter is an ultrafilter, and we are done.  $\square$

In order for the previous theorem to be of any use, we need to exhibit models where  $\diamond(\mathfrak{r}_P)$  holds. Recall that by [9, Thm. 6.6], in many of the models of Set Theory that are obtained via countable support iterations of proper forcing notions, we will have that  $\diamond(\mathfrak{r}_P)$  holds if and only if  $\mathfrak{r}_P = \omega_1$ . The following theorem gives some bounds for this cardinal invariant. First, we consider the cardinal invariant  $\mathfrak{r}_\mathbb{Q}$ , introduced in [1]. This invariant is the reaping number of the Boolean algebra  $\mathcal{P}(\mathbb{Q})/\text{nwd}$ , equivalently, the minimal size of a family of somewhere dense subsets of  $\mathbb{Q}$  such that every set  $A \subseteq \mathbb{Q}$  either contains or is disjoint from an element of the family. It is known [1, Thm. 3.6] that  $\max\{\mathfrak{r}, \text{cof}(\mathcal{M})\} \leq \mathfrak{r}_\mathbb{Q} \leq \mathfrak{i}$ .

**Theorem 2.5.**

$$\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{r}_P \leq \mathfrak{r}_\mathbb{Q}.$$

*Proof.* To see that  $\mathfrak{r} \leq \mathfrak{r}_P$ , note that if  $\mathcal{R}$  is a family of perfect subsets of  $\mathbb{Q}$  witnessing the definition of  $\mathfrak{r}_P$ , and if  $\{U_n : n \in \omega\}$  is a fixed basis for the topology of  $\mathbb{Q}$  consisting of clopen sets, then one can define the family

$$\mathcal{R}' = \{P \cap U_n : P \cap U_n \neq \emptyset\}.$$

Then  $\mathcal{R}'$  is again a family of perfect sets, has the same size as  $\mathcal{R}$ , and is reaping, i.e. for every  $Y \subseteq \mathbb{Q}$  there is a  $P \in \mathcal{R}'$  such that  $P \subseteq Y$  or  $P \cap Y = \emptyset$ . Thus  $\mathcal{R}'$  is a witness to  $\mathfrak{r}$ .

Now let us see that  $\mathfrak{r}_P \geq \mathfrak{d}$ . Let  $\{P_\alpha \mid \alpha < \kappa\}$  be a family of perfect sets, and we will argue that if  $\kappa < \mathfrak{d}$  then this family cannot be reaping. For each  $n < \omega$  let

$I_n = (n\sqrt{2}, (n+1)\sqrt{2})$ , so that  $\mathbb{Q} = \bigcup_{n < \omega} I_n$ . Now enumerate  $I_n = \{q_{n,k} \mid k < \omega\}$ , and for each  $\alpha < \kappa$  we will define  $f_\alpha : \omega \rightarrow \omega$  as follows: For every  $n < \omega$ , assuming we have defined  $f_\alpha(i)$  for all  $i < n$ , we look at the least  $k \geq n$  such that  $I_k \cap P_\alpha \neq \emptyset$  and we let  $j$  be the least such that  $q_{k,j} \in I_k \cap P_\alpha$ . Then we define  $f_\alpha(n) = f_\alpha(n+1) = \dots = f_\alpha(k) = j$ . Now if  $\kappa < \mathfrak{d}$ , we can find an  $f$  that is not dominated by any  $f_\alpha$ . Without loss of generality  $f$  is increasing, so if we let  $A = \{q_{i,j} \mid j \leq f(i)\}$  then for every  $\alpha < \kappa$ , the fact that  $f_\alpha(k) \leq f(k)$  for infinitely many  $k$  implies that  $P_\alpha \cap A$  is infinite, whereas  $A \cap I_n$  is finite for every  $n < \omega$  so  $P_\alpha \setminus A$  must be in  $\mathcal{P}'$  and therefore also be infinite. Hence no  $P_\alpha$  reaps  $A$ .

To see that  $\mathfrak{r}_P \leq \mathfrak{r}_Q$  take a family  $\mathcal{R}$ , of size  $\mathfrak{r}_Q$ , of somewhere dense subsets of  $\mathbb{Q}$  such that every set  $A \subseteq \mathbb{Q}$  is reaped by an element of  $\mathcal{R}$ . For each element  $X \in \mathcal{R}$ , pick a perfect subset  $P_X \subseteq X$ . Then the family  $\mathcal{R}' = \{P_X \mid X \in \mathcal{R}\}$  is a witness to  $\mathfrak{r}_P$ .  $\square$

We now briefly explain how to modify the above construction in order to ensure that our gruff ultrafilter is also a  $\mathbb{Q}$ -point. If we are assuming that  $\diamond(\mathfrak{r}_P)$  holds, then by [9, Prop. 2.5] it follows that  $\mathfrak{r}_P = \omega_1$  and so by Theorem 2.5, we can conclude that  $\mathfrak{d} = \omega_1$ . Now by [2, Thm. 2.10], this means that there is a sequence  $\langle \mathcal{I}_\alpha \mid \alpha < \omega_1 \rangle$  of partitions of  $\omega$  into intervals which is dominating, i.e. for every partition  $\mathcal{J} = \{J_n \mid n < \omega\}$  of  $\omega$  into intervals (we will always assume our partitions into intervals to be ordered increasingly), there exists an  $\alpha < \omega_1$  such that  $\mathcal{J}$  is dominated by  $\mathcal{I}_\alpha$ , which means that if  $\mathcal{I}_\alpha = \{I_\alpha^n \mid n < \omega\}$  then  $(\forall^\infty n < \omega)(\exists k < \omega)(J_k \subseteq I_\alpha^n)$ . It is easily seen that this implies that for every partition  $\mathcal{J} = \{J_n \mid n < \omega\}$  of  $\omega$  into intervals, there is  $\alpha < \omega_1$  such that  $(\forall^\infty n < \omega)(\exists k < \omega)(J_n \subseteq I_\alpha^k \cup I_\alpha^{k+1})$ . This clearly implies that, if  $u$  is an ultrafilter satisfying  $(\forall \alpha < \omega_1)(\exists X \in u)(\forall n < \omega)(|X \cap I_\alpha^n| \leq 1)$ , then  $u$  is a  $\mathbb{Q}$ -point. Thus, if we modify the construction in the proof of Theorem 2.4 to ensure that each  $X_\alpha$  thus constructed satisfies  $(\forall n < \omega)(|X \cap I_\alpha^n| \leq 1)$ , then we will succeed in ensuring that our gruff ultrafilter is also a  $\mathbb{Q}$ -point. But notice that each  $X_\alpha$  is constructed to be a subset of some two-sided perfect set  $X$  whose existence is guaranteed by invoking Lemma 2.3, so all we need to do is to modify the proof of this lemma to ensure that, if we are given a partition  $\mathcal{I} = \{I_n \mid n < \omega\}$  of  $\omega$  into intervals, the two-sided perfect set  $X$  obtained by this the lemma satisfies  $(\forall n < \omega)(|X \cap I_n| \leq 1)$ . This is very easy to do, since the elements of the set  $X$  are successively chosen to be the least-indexed ones satisfying certain condition, so it suffices to also require them to not belong to any  $I_n$  to which some formerly chosen element of  $X$  belongs. It follows that, by introducing small modifications in the proofs of Lemma 2.3 and Theorem 2.4, we can construct a gruff ultrafilter that is at the same time a  $\mathbb{Q}$ -point.

**Corollary 2.6.** *There are gruff  $\mathbb{Q}$ -points in the Sacks model.*

*Proof.* Since  $\mathfrak{r}_Q = \mathfrak{i} = \omega_1$  in Sacks model, Theorem 2.5 together with Theorem 2.1 implies that  $\diamond(\mathfrak{r}_P)$  holds in this model, so the result follows from Theorem 2.4 together with the observations outlined on the previous paragraph.  $\square$

**Question 2.7.** *Is either of the two inequalities in Theorem 2.5 consistently strict?*

### 3. THERE ARE GRUFF ULTRAFILTERS IF $\mathfrak{d} = \mathfrak{c}$

In this section we will obtain the existence of gruff ultrafilters from the cardinal invariant assumption that  $\mathfrak{d} = \mathfrak{c}$ . In order to do this, we will first lay down some

notation and preliminary results that are central to our proof, and that will also be relevant also for the next section.

We describe a method that, given a function  $f : \omega \rightarrow \omega$ , allows us to use  $f$  to construct a perfect subset of every  $X \in \mathcal{P}$ . From now on we will fix an effective enumeration  $\{q_n \mid n < \omega\}$  of  $\mathbb{Q}$ . Now, given  $f : \omega \rightarrow \omega$ , define the following clopen subsets of  $\mathbb{Q}$ :

$$J_n^f = \left( q_n - \frac{\sqrt{2}}{k}, q_n + \frac{\sqrt{2}}{k} \right),$$

where  $k$  is the least possible natural number that ensures  $q_m \notin J_n^f$  for every  $n \neq m \leq f(n)$ . Thus  $J_n^f$  is a clopen interval, centred at  $q_n$ , which is just barely small enough so that it does not contain any of the finitely many  $q_m$  with  $m \neq n$  and  $m \leq f(n)$ . Clearly, the faster the function  $f$  grows, the smaller the interval  $J_n^f$  will be. In other words, if  $f(n) \geq g(n)$  then  $J_n^f \subseteq J_n^g$ . Note also that  $J_n^f$  is completely determined by the single value  $f(n)$ .

Now, given any subset  $X \subseteq \mathbb{Q}$ , we define

$$X(f) = \mathbb{Q} \setminus \left( \bigcup_{q_n \notin X} J_n^f \right) \subseteq X.$$

Then  $X(f)$  is a closed set, since it is the complement of an open set. Some properties of the sets  $X(f)$  that we will need later on, and that are easy to check, are the following:

- If  $X \subseteq Y$ , then  $X(f) \subseteq Y(f)$ ,
- $X(f) \cap Y(f) = (X \cap Y)(f)$ , and similarly for intersections of any finite amount of sets,
- If  $g \leq f$ , then  $X(g) \subseteq X(f)$ ,
- If  $f$  is unbounded, then every rational number belongs to only finitely many of the  $J_n^f$ . In fact, if  $f$  is strictly increasing then the rational number  $q_n$  can only belong to at most  $n$  of the  $J_m^f$ , since in this case, whenever  $m > n$  we have that  $n \leq f(n) < f(m)$  and therefore  $q_n \notin J_m^f$ .

We wish the set  $X(f)$  to be not only closed, but also perfect. Of course, we can only hope to achieve this when  $X \in \mathcal{P}$ .

**Lemma 3.1.** *For every crowded unbounded subset  $X \subseteq \mathbb{Q}$ , there exists a function  $f_X$  such that, whenever  $g$  is an increasing function with  $g \not\leq^* f_X$ ,  $X(g)$  is crowded unbounded (and hence perfect unbounded).*

*Proof.* Recursively define  $f_X$  by  $f_X(0) = \min\{k < \omega \mid k > 0 \text{ and } q_k \in X\}$ ; and once  $f_X(n-1)$  has been defined, for every  $m < n$  with  $q_m \in X$  we define the sets

$$A_n^m = \{k < \omega \mid f_X(n-1) < k, q_k \in X, |q_m - q_k| < \frac{1}{2^n} \text{ and } q_k \notin \bigcup_{m < l < n} J_l^{\text{id}}\},$$

where  $\text{id} : \omega \rightarrow \omega$  is the identity function. We also define the set

$$A_n = \{k < \omega \mid f_X(n-1) < k, q_k \in X, q_k > n \text{ and } q_k \notin \bigcup_{l < n} J_l^{\text{id}}\}.$$

The  $A_n^m$  are nonempty because  $X$  is crowded, and  $A_n$  is nonempty because  $X$  is unbounded. So we can let

$$f_X(n) = \max(\{\min(A_n^m) \mid m < n \text{ and } q_m \in X\} \cup \{\min(A_n)\}).$$

This finishes the definition of  $f_X$ . Now let  $g$  be an increasing function such that  $g \not\leq^* f_X$ . Note that, since  $g$  is increasing, we have  $\text{id} \leq g$ , and so  $J_n^g \subseteq J_n^{\text{id}}$  for every  $n < \omega$ . To prove that  $X(g)$  is unbounded, let  $m \in \mathbb{N}$ , and we will find an element  $x \in X(g)$  with  $x > m$ . To that end, we fix an  $n > m$  such that  $f_X(n) < g(n)$ . Letting  $k = \min(A_n)$ , by definition we have that  $x = q_k \in X$ ,  $q_k > n > m$  and  $q_k \notin \bigcup_{l < n} J_l^{\text{id}}$ , and therefore  $q_k \notin \bigcup_{l < n} J_l^g$ . Now since  $k \leq f_X(n) < g(n)$ , we also have that  $q_k \notin J_l^g$  for  $l \geq n$ , as long as  $l \neq k$ . But in particular,  $q_k \notin J_l^g$  whenever  $q_l \notin X$ , thus  $x = q_k \in X(g)$ .

Now to prove that  $X(g)$  is crowded, we pick an arbitrary  $q_m \in X(g)$  and an  $\varepsilon > 0$ , and try to find some  $x \in X(g)$  with  $x \neq q_m$  and  $|x - q_m| < \varepsilon$ . Note that, since  $q_m \in X(g)$ , then we must have in particular that  $q_m \notin J_l^g$  whenever  $l < m$ . We let  $N$  be large enough that  $\frac{1}{2^N} < \varepsilon$  and  $(q_m - \frac{1}{2^N}, q_m + \frac{1}{2^N}) \cap J_l^g = \emptyset$  for all  $l < m$ . We now pick an  $n > N$  such that  $f_X(n) < g(n)$ , and let  $k = \min(A_n^m)$ . Then by definition we have that  $x = q_k \in X$ ,  $|q_m - q_k| < \frac{1}{2^n} < \varepsilon$ , and  $q_k \notin J_l^{\text{id}} \supseteq J_l^g$  for all  $m < l < n$ . Also, since  $k \leq f_X(n) < g(n)$ , we will have  $q_k \notin J_l^g$  for all  $l \geq n$ , as long as  $l \neq k$ . Moreover, our choice of  $n$  (and  $N$ ) also ensures that  $q_k \notin J_l^n$  for  $l < m$ . In other words, the only cases where we might have  $q_k \in J_l^n$  would be when  $l$  is either equal to  $m$ , or equal to  $k$ ; but since  $q_k, q_m \in X$ , we conclude that  $k = q_k \in X(g)$ .  $\square$

This lemma will be of key importance for the rest of this article. We will first use it for proving the following result, which strengthens both the theorem of Copláková and Hart (which assumes  $\mathfrak{b} = \mathfrak{c}$ ), and the one of van Douwen (which assumes  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ ).

**Theorem 3.2.** *If  $\mathfrak{d} = \mathfrak{c}$ , then there exists a gruff ultrafilter.*

*Proof.* Let  $\{A_\alpha \mid \alpha < \mathfrak{c}\}$  be an enumeration of all subsets of  $\mathbb{Q}$ . We will recursively construct sets  $X_\alpha \subseteq \mathbb{Q}$ , satisfying the following conditions for every  $\alpha < \mathfrak{c}$ :

- (1)  $X_\alpha$  is a perfect unbounded subset of  $\mathbb{Q}$ ,
- (2)  $X_\alpha$  is either contained in or disjoint from  $A_\alpha$ , and
- (3) the family  $\{X_\xi \mid \xi \leq \alpha\}$  generates a filter all of whose elements belong to  $\mathcal{P}$  (i.e. they contain a crowded unbounded set).

So suppose that we have already constructed  $\{X_\xi \mid \xi < \alpha\}$  satisfying all three conditions. Since  $\mathcal{P}$  is a coideal, we can choose an  $A \in \{A_\alpha, \mathbb{Q} \setminus A_\alpha\}$  such that the family  $\{X_\xi \mid \xi < \alpha\} \cup \{A\}$  generates a filter with all elements belonging to  $\mathcal{P}$ . If we now let  $B$  be the maximal crowded unbounded subset of  $A$  (this  $B$  can be seen as either the union of all crowded subsets of  $A$ , or the part of  $A$  that remains after performing the Cantor-Bendixson process), we can see that  $\{X_\xi \mid \xi < \alpha\} \cup \{B\}$  still generates a filter all of whose elements belong to  $\mathcal{P}$ .

For each finite  $F \subseteq \{X_\xi \mid \xi < \alpha\}$ , we take the maximal crowded unbounded subset  $B_F \subseteq (\bigcap F) \cap B$  (since by hypothesis, the latter set belongs to  $\mathcal{P}$ ), and consider the function  $f_{B_F} : \omega \rightarrow \omega$  as given by Lemma 3.1. Since there are  $\max\{|\alpha|, \omega\} < \mathfrak{c}$  possible  $F \subseteq \{X_\xi \mid \xi < \alpha\}$ , by  $\mathfrak{d} = \mathfrak{c}$  we can choose a  $g : \omega \rightarrow \omega$  such that for each of the  $F$ , we have  $g \not\leq^* f_{B_F}$ . We now let  $X_\alpha = B(g)$ . Since in particular  $g \not\leq^* f_B$  (since  $B = B_\emptyset$ ),  $X_\alpha$  will be a perfect unbounded subset of  $B$ , and for every finite

$F \subseteq \{X_\xi \mid \xi < \alpha\}$ , we notice that

$$X_\alpha \cap \left( \bigcap_{\xi \in F} X_\xi \right) \supseteq B(g) \cap \left( \bigcap_{\xi \in F} X_\xi(g) \right) = \left( B \cap \left( \bigcap_{\xi \in F} X_\xi \right) \right) (g) \supseteq B_F(g),$$

and the rightmost set above is crowded unbounded. Thus our  $X_\alpha$  can be added to the previously obtained  $X_\xi$  while preserving the three required properties, and so the recursive construction can continue. In the end we let  $u$  be the filter generated by  $\{X_\alpha \mid \alpha < \mathfrak{c}\}$ , which will clearly be a gruff ultrafilter.  $\square$

#### 4. GRUFF ULTRAFILTERS IN THE RANDOM REAL MODEL

We will now proceed to prove that there are gruff ultrafilters in the Random real model. We will closely follow a strategy used by Paul E. Cohen (not to be confused with the Paul J. Cohen that discovered forcing!) in his proof that there are P-points in the Random model [5]. For this purpose, we introduce the following definition.

**Definition 4.1.** A continuous increasing  $\omega_1$ -sequence  $\langle A_\alpha \mid \alpha < \omega_1 \rangle$  of reals (elements of  $\omega^\omega$ ) will be called a **strong pathway** if it satisfies the following three properties:

- (1)  $\bigcup_{\alpha < \omega_1} A_\alpha = \omega^\omega$ ,
- (2) for every  $\alpha < \beta < \omega_1$  there exists a  $g \in A_\beta$  which is not dominated by any  $f \in A_\alpha$ .
- (3) for every  $\alpha < \omega_1$ , if  $f_1, \dots, f_n \in A_\alpha$  and  $f \in \omega^\omega$  is definable from  $f_1, \dots, f_n$ , then  $f \in A_\alpha$  (i.e.  $A_\alpha$  is closed under set-theoretic definability),

Cohen [5] defined a pathway to be a continuous, increasing  $\omega_1$ -sequence satisfying requirements (1) and (2) from Definition 4.1, along with the requirement (strictly weaker than (3) in the aforementioned Definition) that each  $A_\alpha$  is closed under joins (the join of  $f, g$  is the function mapping  $2n$  to  $f(n)$  and  $2n+1$  to  $g(n)$ ) and Turing-reducibility. In other words, in a pathway, we can only ensure that  $f \in A_\alpha$  if it is explicitly (algorithmically) computable (in finitely many steps) from  $f_1, \dots, f_n \in A_\alpha$ . Since this weak closure will not be enough for our purposes, we adapted Cohen's definition by demanding closure under any set-theoretic construction.

Now our proof will be done in two main steps. The first step is proving that there is a strong pathway in the Random real model, and the second step is showing that the existence of a strong pathway implies the existence of a gruff ultrafilter. Each of the following two theorems realizes each of these two steps.

**Theorem 4.2.** *Take a ground model  $\mathbf{V}$  that satisfies CH, let  $\lambda$  be a cardinal in  $\mathbf{V}$ , and let  $\mathcal{R}_\lambda$  be the forcing that adds  $\lambda$  many Random reals (this is,  $\mathcal{R}_\lambda$  consists of all non-null sets in the measure algebra of  $2^\lambda$ , ordered by inclusion modulo null sets). If  $r : \lambda \rightarrow 2$  is the sequence of Random reals added by  $\mathcal{R}_\lambda$  (this is, if  $r$  is the unique element that belongs to the intersection of an  $(\mathcal{R}_\lambda, \mathbf{V})$ -generic filter), then in  $\mathbf{V}[r]$  there exists a strong pathway.*

*Proof.* We will start working in the ground model  $\mathbf{V}$ . Since  $\mathbf{V} \models \text{CH}$ , we can fix an enumeration  $\langle f_\alpha \mid \alpha < \omega_1 \rangle$  of  $\omega^\omega$ . We now choose a large enough cardinal  $\theta$ , and construct a continuous increasing  $\omega_1$ -chain of countable elementary submodels  $M_0 \prec M_1 \prec \dots \prec M_\alpha \prec \dots \prec H(\theta)$  satisfying, for every  $\alpha < \omega_1$ , that  $f_\alpha \in M_{\alpha+1}$

and also that some  $f \in M_{\alpha+1}$  dominates all elements of  $\omega^\omega \cap M_\alpha$ . We note that, if  $\pi : \omega \rightarrow \lambda$  is an injective function from  $\mathbf{V}$ , then  $r \circ \pi : \omega \rightarrow 2$  is a Random real over  $\mathbf{V}$  (since it will avoid any Borel null set whose code belongs to  $\mathbf{V}$ ), and hence also over each of the  $M_\alpha$  (since Borel codes that belong to  $M_\alpha$  also belong to  $\mathbf{V}$ , and by elementarity). We therefore define the  $A_\alpha$  to be given by

$$A_\alpha = \bigcup_{\substack{\pi: \omega \rightarrow \lambda \\ \pi \in \mathbf{V} \text{ is injective}}} \omega^\omega \cap M_\alpha[r \circ \pi].$$

We now proceed to prove that the sequence  $\langle A_\alpha \mid \alpha < \omega_1 \rangle$  (which clearly is a continuous increasing sequence) satisfies all three requirements of the definition of a strong pathway.

- (1) Given a real  $f \in \mathbf{V}[r]$ , the structure of  $\mathcal{R}_\lambda$  as a measure algebra, along with the fact that it is a c.c.c. forcing notion, imply that  $f$  can be computed from countably many bits from  $r$ , along with countably many reals. In other words, there are  $f_0, f_1, \dots, f_n, \dots \in \mathbf{V}$  and there is an injection  $\pi : \omega \rightarrow \lambda$  such that, if  $\alpha < \omega_1$  is large enough that all  $f_i \in M_\alpha$ , then  $f \in M_\alpha[r \circ \pi]$ . This proves that  $\bigcup_{\alpha < \omega_1} A_\alpha = \omega^\omega$ .
- (2) Let  $\alpha < \beta < \omega_1$  and  $\pi : \omega \rightarrow \lambda$  be an injection. Since  $\mathcal{R}_\lambda$  is an  $\omega^\omega$ -bounding forcing notion, every element of  $M_\alpha[r \circ \pi]$  (which is a Random extension of  $M_\alpha$ ) is dominated by some real in  $M_\alpha$ , and all reals in  $M_\alpha$  are in turn dominated by a single real  $g \in M_\beta$ . Hence this  $g$  dominates all elements of  $A_\alpha$  (in particular,  $g$  is not dominated by any  $f \in A_\alpha$ ).
- (3) Let  $f$  be set-theoretically definable from  $f_1, \dots, f_n \in A_\alpha$ . There are injective functions  $\pi_i : \omega \rightarrow \lambda$ , for  $1 \leq i \leq n$ , such that  $f_i \in M_\alpha[r \circ \pi_i]$ . For each nonempty subset  $a \subseteq n$ , define

$$\Lambda_a = \left( \bigcap_{i \in a} \text{ran}(f_i) \right) \setminus \left( \bigcup_{j \notin a} \text{ran}(f_j) \right) \subseteq \lambda.$$

Note that the  $\Lambda_a$  are all countable, and pairwise disjoint. Now, working in  $\mathbf{V}$ , pick an effective partition of  $\omega$  into  $2^n - 1$  cells  $N_a$  (indexed by the nonempty subsets  $a \subseteq n$ ), with each  $N_a$  of cardinality  $|\Lambda_a|$ , and let  $\pi : \omega \rightarrow \lambda$  be an injection which bijectively maps each  $N_a$  onto  $\Lambda_a$ . Since the  $\Lambda_a$  are definable, they inhabit  $M_\alpha$ , and so do the sets  $R_i = \bigcup_{i \in a} N_a$ , which allow us to reconstruct  $r \circ \pi_i$  from  $r \circ \pi$  (note that  $\pi[R_i] = \text{ran}(f_i)$ ). Thus for each  $i$  we have that

$$f_i \in M_\alpha[r \circ \pi_i] \subseteq M_\alpha[r \circ \pi],$$

and therefore it must be the case that  $f \in M_\alpha[r \circ \pi] \subseteq A_\alpha$ , so  $A_\alpha$  has the required closure, and we are done.  $\square$

In the proof of the following theorem, we shall be using the same effective enumeration  $\{q_n \mid n < \omega\}$  of  $\mathbb{Q}$  that we used in the previous section, along with the definitions of the  $J_n^f$  and  $X(f)$  for  $n < \omega$ ,  $f : \omega \rightarrow \omega$ , and  $X \subseteq \mathbb{Q}$ . Also, given a set  $X \in \mathcal{P}$ , we will occasionally invoke the function  $f_X$  whose existence is guaranteed by Lemma 3.1. Finally, we will use the fact that for every  $X \subseteq \mathbb{Q}$ , if  $Y$  is the maximal crowded subset of  $X$  ( $Y$  can be defined to be the union of all crowded

subsets of  $X$ , or alternatively, the end result of the transfinite Cantor-Bendixson decomposition of  $X$ ), then the function  $\chi_{\{n < \omega \mid q_n \in Y\}}$  is definable from  $\chi_{\{n < \omega \mid q_n \in X\}}$ .

**Theorem 4.3.** *Let  $\lambda$  be any infinite cardinal number in a ground model  $\mathbf{V}$  that satisfies CH, and let  $\mathcal{R}_\lambda$  be the forcing notion that adds  $\lambda$  many Random reals to  $\mathbf{V}$ . If  $r : \lambda \rightarrow 2$  is the generic function added by  $\mathcal{R}_\lambda$ , then in  $\mathbf{V}[r]$  there is a gruff ultrafilter.*

*Proof.* By Theorem 4.2, we can let  $\langle A_\alpha \mid \alpha < \omega_1 \rangle$  be a strong pathway in  $\mathbf{V}[r]$ . For each  $\alpha$ , choose a function  $g_\alpha \in A_{\alpha+1}$  which is not dominated by any element of  $A_\alpha$ . We will recursively construct a continuous increasing sequence of filters  $\mathcal{F}_\alpha \subseteq \mathcal{P}$ , satisfying the following for every  $\alpha < \omega_1$ :

- (1)  $\mathcal{F}_\alpha$  has a basis of perfect unbounded sets  $P \subseteq \mathbb{Q}$  such that

$$\chi_{\{n < \omega \mid q_n \in P\}} \in A_\alpha,$$

- (2) for every  $X \subseteq \mathbb{Q}$  such that  $\chi_{\{n < \omega \mid q_n \in X\}} \in A_\alpha$ , either  $X$  or  $\mathbb{Q} \setminus X$  belongs to  $\mathcal{F}_{\alpha+1}$ .

If we succeed in performing such a construction, then clearly  $u = \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$  will be an ultrafilter (by condition (2)), with a basis of perfect unbounded sets (by condition (1)), i.e. a gruff ultrafilter.

We start off with  $\mathcal{F}_0 = \{\mathbb{Q}\}$ , and when  $\alpha$  is a limit ordinal we let  $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$ . We only need to show that the  $\mathcal{F}_\alpha$  thus defined satisfies condition (1) (note that condition (2) is vacuous in this case, since it only concerns successor cardinals). To this effect, let  $X \in \mathcal{F}_\alpha$ , which means that  $X \in \mathcal{F}_\beta$  for some  $\beta < \alpha$ . Now by our inductive hypothesis (1), there exists some  $P \in \mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$  such that  $P \subseteq X$  and  $\chi_{\{n < \omega \mid q_n \in P\}} \in A_\beta \subseteq A_\alpha$ , which shows that condition (1) still holds of  $\mathcal{F}_\alpha$ .

Now assume that we have already constructed  $\mathcal{F}_\alpha$  satisfying conditions (1) and (2). In order to construct  $\mathcal{F}_{\alpha+1}$ , we first extend  $\mathcal{F}_\alpha$  to some ultrafilter  $\mathcal{G} \subseteq \mathcal{P}$ , and let  $\mathcal{F}_{\alpha+1}$  be the upward closure of the family

$$\{X(g_\alpha) \mid X \in \mathcal{G}, \chi_{\{n < \omega \mid q_n \in X\}} \in A_\alpha \text{ and } X \text{ is crowded unbounded}\}.$$

First of all, notice that each of the  $X(g_\alpha)$  generating  $\mathcal{F}_{\alpha+1}$  is a perfect unbounded set (hence  $\mathcal{F}_{\alpha+1} \subseteq \mathcal{P}$ ), since  $\chi_{\{n < \omega \mid q_n \in X\}} \in A_\alpha$  implies that the  $f_X$  given by Lemma 3.1 is also an element of  $A_\alpha$ , hence  $g_\alpha \not\leq^* f_X$  and so  $X(g_\alpha)$  is crowded unbounded. This reasoning also helps us establish that  $\mathcal{F}_{\alpha+1}$  is indeed a filter, since whenever  $Y_1, \dots, Y_n \in \mathcal{F}_{\alpha+1}$  it is because for some crowded unbounded  $X_1, \dots, X_n \in \mathcal{G}$  with  $\chi_{\{n < \omega \mid q_n \in X_i\}} \in A_\alpha$ , we have that  $X_i(g_\alpha) \subseteq Y_i$ . But if we let  $X$  be the maximal crowded unbounded subset of  $X_1 \cap \dots \cap X_n$  (which exists since this intersection belongs to  $\mathcal{G} \subseteq \mathcal{P}$ ), then it will be the case that  $X \in \mathcal{G}$ ,  $\chi_{\{n < \omega \mid q_n \in X\}} \in A_\alpha$ , and therefore

$$Y_1 \cap \dots \cap Y_n \supseteq X_1(g_\alpha) \cap \dots \cap X_n(g_\alpha) = (X_1 \cap \dots \cap X_n)(g_\alpha) \supseteq X(g_\alpha) \in \mathcal{F}_{\alpha+1}.$$

Now, in order to show that  $\mathcal{F}_{\alpha+1}$  extends  $\mathcal{F}_\alpha$ , we let  $X \in \mathcal{F}_\alpha$ . By condition (1) on our induction hypothesis, there exists a perfect set  $P \subseteq X$  such that  $\chi_{\{n < \omega \mid q_n \in P\}} \in A_\alpha$  and  $P \in \mathcal{F}_\alpha$ . Therefore we have that  $X \supseteq P \supseteq P(g_\alpha) \in \mathcal{F}_{\alpha+1}$ , and so  $\mathcal{F}_\alpha \subseteq \mathcal{F}_{\alpha+1}$ .

We now notice also that all of the  $X(g_\alpha)$  that generate the filter  $\mathcal{F}_{\alpha+1}$  are such that  $\chi_{\{n < \omega \mid q_n \in X(g_\alpha)\}} \in A_{\alpha+1}$ , since each  $X(g_\alpha)$  is definable from  $\chi_{\{n < \omega \mid q_n \in X\}} \in A_\alpha \subseteq A_{\alpha+1}$  and  $g_\alpha \in A_{\alpha+1}$  (and our fixed enumeration of  $\mathbb{Q}$ , which lies in  $A_0 \subseteq A_{\alpha+1}$ ). Hence our  $\mathcal{F}_{\alpha+1}$  satisfies condition (1) of the construction.

To show that condition (2) is also satisfied, let  $X \subseteq \mathbb{Q}$  be such that  $\chi_{\{n < \omega \mid q_n \in X\}} \in A_\alpha$ , and let us show that  $\mathcal{F}_{\alpha+1}$  contains one of  $X, \mathbb{Q} \setminus X$ . Since  $\mathcal{G}$  is an ultrafilter, there is  $Y \in \{X, \mathbb{Q} \setminus X\}$  such that  $Y \in \mathcal{G}$ . Since  $\mathcal{G} \subseteq \mathcal{P}$ , if we let  $Z$  be the maximal crowded unbounded subset of  $Y$ , then  $Z \in \mathcal{G}$ . But notice that  $\chi_{\{n < \omega \mid q_n \in Z\}} \in A_\alpha$ , and this implies that  $Y \supseteq Z \supseteq Z(g_\alpha) \in \mathcal{F}_{\alpha+1}$ .

This finishes the recursive construction, and we are done.  $\square$

The authors would like to thank Andreas Blass for some very fruitful discussions, particularly concerning Cohen's paper [5], as well as for greatly simplifying the proof of Lemma 1.2.

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