HINDMAN'S THEOREM IN THE HIERARCHY OF CHOICE PRINCIPLES

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Abstract. In the context of ZF, we analyze a version of Hindman's finite unions theorem on infinite sets, which normally requires the Axiom of Choice to be proved. We establish the implication relations between this statement and various classical weak choice principles, thus precisely locating the strength of the statement as a weak form of the AC.

1. Introduction

One of the central results of infinitary (countable) Ramsey theory is the so-called Hindman's finite sums theorem [9], stating that for every finite partition of $\mathbb{N}$ it is possible to find elements $x_1 < \cdots < x_n < \cdots$ such that all sums of finitely many of the x_i, with no repetitions, are contained in the same cell of the partition. An extremely close result in a similar vein, which was in fact already known to be equivalent to Hindman's finite sums theorem before the latter was proved, is the statement that for every partition of the set $[\mathbb{N}]^{<\omega}$ of all finite subsets of $\mathbb{N}$, one can find infinitely many pairwise disjoint sets such that all unions of finitely many of them are contained within the same cell of the partition. Upon replacing $\mathbb{N}$ with an arbitrary set X in the latter result, one obtains a statement that, while provable in ZFC, may potentially not be a theorem of ZF. This statement is what we will refer to as *Hindman's theorem* in this paper, and it will be our central object of study.

Definition 1.1. *Hindman's theorem*, denoted HT, is the statement that, for every infinite set X and for every colouring $c : [X]^{<\omega} \longrightarrow 2$ of the finite powerset of X with two colours, there exists an infinite, pairwise disjoint family $Y \subseteq [X]^{<\omega}$ such that the set

$$\mathrm{FU}(Y) = \left\{ \bigcup_{y \in F} y \middle| F \in [Y]^{<\omega} \setminus \{\varnothing\} \right\}$$

is c-monochromatic.

(We prove later, in Proposition 2.1, that we obtain an equivalent statement, modulo ZF, if we vary the number of colours in the colouring, so long as said number remains finite.) It follows from Hindman's finite unions theorem over $\mathbb{N}$ that HT is a theorem of ZFC (by simply embedding $\mathbb{N}$ into any infinite set X and restricting any colouring of $[X]^{<\omega}$); however, it turns out that one cannot prove HT in ZF only. Hence, one can think of the statement HT as a weak form of the Axiom of Choice, and it thus makes sense to try and compare this choice principle with other classical choice principles that have been extensively studied, investigating the implication relations (modulo ZF) that there are between them. It is worth noting that HT is a very natural choice principle not only due to its origins in Ramsey theory, but also in light of some results presented in this paper, e.g., Proposition 2.3, stating that the conjunction of HT and König's Lemma is equivalent to the statement that every infinite set is Dedekind-infinite (and therefore, the latter is also equivalent to the conjunction of HT and Ramsey's theorem). It is also worth noting that HT is equivalent to a statement that simply deals with the Dedekind-finiteness of finite powersets of

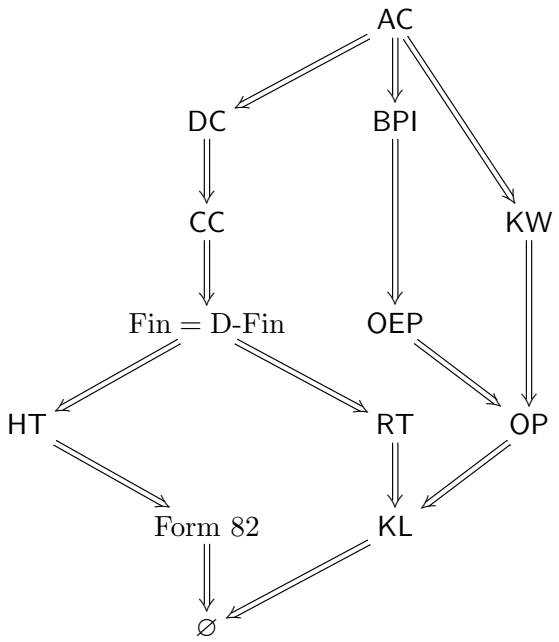

FIG. 1. Implications between HT and other classical choice principles.

sets (see Proposition 2.2 below), suggesting that the algebraic and topological dynamics aspects of Hindman's Theorem are not particularly relevant from the point of view of choice principles.

In this paper, we locate the precise strength of HT among the most important classical choice principles[1]. The choice principles considered are, in addition to the Axiom of Choice, the principle of Countable Choice, the axiom of Dependent Choice, König's Lemma, the principle that every Dedekind-finite set is finite, the Boolean Prime Ideal theorem, the Kinna–Wagner selection principle, the Ordering Principle, the Order Extension Principle, Ramsey's Theorem, and Form 82 from [10] (the latter is not as classical as the other ones, but we include it in our study due to its high degree of similarity with a certain equivalence of HT). The results we obtain are summarized in the diagram from Fig. 1, which contains all possible ZF-provable implications between HT and the aforementioned choice principles, each of which is represented by the obvious abbreviation in the diagram. Formal definitions of each of the choice principles considered are to be found in Sec. 2.

The reader will note that the diagram from Fig. 1 contains very few implication arrows to and from HT (the only ones are Fin = D-Fin $\Rightarrow$ HT and HT $\Rightarrow$ Form 82, both of which will be obvious, given the equivalence of HT established in Proposition 2.2, once we state the meaning of the involved choice principles). Therefore, we must emphasize that the main body of work presented in this paper is not proofs of implications in ZF, but rather *independence proofs*, showing that there are no further implications between HT and any other of the principles mentioned. In other words, the most meaningful information that can be gathered from the diagram in Fig. 1 is not the arrows shown, but rather the ones *not* shown, signalling that an independence proof (or an argument stemming from a previous independence proof) has been established formally. As such, most of the content of this paper deals either with symmetric models, or with Fraenkel–Mostowkski permutation models (which yield symmetric models after applying well-known transfer theorems), thus obtaining models of ZF witnessing the unprovability of the relevant statement. In Sec. 2 we relay a few basic

[1]Further work of E. Tachtsis [17] has established some equivalences between HT and other previously known weak choice principles.

ZF results; afterwards in Sec. 3 we discuss Fraenkel–Mostowski permutation models and the transfer theorems that allow us to obtain ZF models from them, and proceed to determine whether HT holds in various of these models. With this information in hand, we establish all the independence results required to complete the diagram in Fig. 1, except for the reversibility of the implication HT $\Rightarrow$ Form 82; this is addressed in Sec. 4, which contains the proof that this implication is not reversible (and the proof is involved enough that it warrants its own section). Finally, in Sec. 5, we consider a weaker Boolean version of HT and also determine its place within the hierarchy of choice principles (see the enhanced diagram from Fig. 2).

2. Some basic results

We begin by establishing that, in our definition of HT, we could have considered colourings on any finite number of colours and still obtained an equivalent statement modulo ZF. Therefore, we will temporarily use the symbol HT(k) (where $k \in \mathbb{N} \setminus \{1\}$) to denote that, for every infinite set X and every colouring $c : [X]^{<\omega} \longrightarrow k$, there exists an infinite, pairwise disjoint $Y \subseteq [X]^{<\omega}$ such that FU(Y) is c-monochromatic. Hence, HT(2) is exactly what we called HT in Definition 1.1; after the following proposition, we will be able to drop the parameter k and simply write HT in all cases.

Proposition 2.1. *All of the statements* HT(k)*, as* $k \in \mathbb{N} \setminus \{1\}$ *varies, are equivalent under* ZF*.*

Proof. Since k-colourings are always also k'-colourings whenever $k \leq k'$, we have that HT(k') $\Rightarrow$ HT(k) under these circumstances. Now to finish the proof, we need only show that HT(k) $\Rightarrow$ HT(k+1) for $k \geq 2$ (which yields an argument by induction). So suppose that $k \geq 2$ and that HT(k) holds. Let X be an infinite set and let $c : [X]^{<\omega} \longrightarrow k+1$ be a colouring. Define another colouring $d : [X]^{<\omega} \longrightarrow k$ by letting $d(x) = \min\{c(x), k-1\}$. Using HT(k) we obtain an infinite pairwise disjoint family $Y \subseteq [X]^{<\omega}$ such that FU(Y) is d-monochromatic, say on colour $i < k$. If $i < k-1$ then FU(Y) is c-monochromatic as well (on the same colour) and we are done; otherwise we know that for every $y \in \mathrm{FU}(Y)$, $c(y) \in \{k-1, k\}$. Hence, we can define yet another colouring $e : [Y]^{<\omega} \longrightarrow 2$ given by $e(F) = k - c\left(\bigcup_{y\in F} y\right)$ and use HT(2) to obtain an infinite, pairwise disjoint family $W \subseteq [Y]^{<\omega}$ such that FU(W) is e-monochromatic, say in colour $j < 2$. This means that, for every $\mathcal{F} \in [W]^{<\omega}$,

$$j = e\left(\bigcup_{F\in\mathcal{F}} F\right) = k - c\left(\bigcup_{y\in\bigcup_{F\in\mathcal{F}} F} y\right),$$

so that, if we define

$$Z = \left\{ \bigcup_{y\in F} y \middle| F \in W \right\},$$

then $Z \subseteq \mathrm{FU}(Y) \subseteq [X]^{<\omega}$ is an infinite, pairwise disjoint family such that FU(Z) is monochromatic for c (in colour $k-j$), and we are done. □

The "classical" choice principles considered in this paper, in addition to HT, are the following:

(1) The *Axiom of Dependent Choice*, abbreviated DC, is the statement that, for every set X equipped with a relation $R \subseteq X \times X$ such that $(\forall x \in X)(\exists y \in X)(x\ R\ y)$, there exists a countable sequence $\langle x_n | n < \omega\rangle$ such that $(\forall n < \omega)(x_n\ R\ x_{n+1})$ (this statement is labelled Form 43 in [10]).

(2) The *Axiom of Countable Choice*, which we will abbreviate CC, is the statement that every countable family of nonempty sets admits a choice function (Form 8 from [10]).

(3) The statement "every infinite set is Dedekind-infinite" will be denoted by Fin = D-Fin (Form 9 in [10]).

(4) *Ramsey's theorem*, denoted by RT, is the statement that for every infinite set X and for every colouring $c : [X]^2 \longrightarrow 2$, there exists an infinite $Y \subseteq X$ such that $[Y]^2$ is c-monochromatic (Form 17 from [10]).

(5) *König's lemma*, which we will abbreviate KL, is the statement that every countable family of nonempty finite sets admits a choice function (Form 10 in [10]).

(6) The *Boolean Prime Ideal theorem*, denoted by BPI, is the statement that every Boolean algebra carries a prime ideal (Form 14 from [10]).

(7) The *Kinna–Wagner selection principle*, which will be abbreviated by KW, is the statement that[2] for every set X there exists an ordinal number α and an injective function $f : X \longrightarrow \wp(\alpha)$ (Form 15 in [10]).

(8) The *Ordering Principle*, denoted by OP, is the statement that every set can be linearly ordered (Form 30 from [10]).

(9) The *Order Extension Principle*, abbreviated OEP, is the statement that every partial order can be extended to a linear order on the same set (Form 49 in [10]).

(10) Form 82 (according to the numbering in [10]) is the statement that for every infinite set X, its powerset $\wp(X)$ is Dedekind-infinite.

Recall that a set is said to be *Dedekind-infinite* if ω injects into it (equivalently, if there exists an injective, but not surjective, function of the set into itself), and a set is *Dedekind-finite* if it is not Dedekind-infinite. It is hard not to see that every Dedekind-infinite set must be infinite; however, the converse to this statement is not provable in ZF, and is therefore considered a choice principle. König's Lemma owes its name to the fact that it is equivalent, over ZF, to the classical theorem about finitely branching infinite trees due to König (that is, the statement that every finitely branching infinite tree must have an infinite branch). Well-known classical results in choiceless set theory establish that, over ZF, DC implies CC which in turn implies Fin = D-Fin; BPI implies OEP, and either OEP or KW implies OP which in turn implies KL. Moreover, Fin = D-Fin implies both Form 82 and RT, and the latter in turn implies KL. Furthermore, none of the implications mentioned in this paragraph is reversible, and there are no further implication relations between any of the choice principles mentioned in this paragraph (see, e.g., [10] for a complete set of references on all the facts just mentioned).

We now begin to analyse the strength of HT among all of these principles. Important information can be gathered by "locally" studying those sets for which Hindman's finite unions theorem holds (as opposed to the "global" principle that every infinite set satisfies Hindman's theorem). Such a careful study was performed in [2], where the following definition is stated.

Definition 2.1 ([2], Definition 3.6 (3), cf. Definition 3.1)**.** A set X will be called *H-finite* if there exists a colouring $c : [X]^{<\omega} \longrightarrow 2$ such that for no infinite, pairwise disjoint $Y \subseteq [X]^{<\omega}$ can the set $\mathrm{FU}(Y)$ be c-monochromatic. We will say that X is *H-infinite* if it is not H-finite (so X is H-infinite if and only if Hindman's finite unions theorem holds at X).

[2]Equivalently, for every family of sets $\mathscr{F}$ all of which have at least two elements, there is a function f with domain $\mathscr{F}$ such that $(\forall S \in \mathscr{F})(\varnothing \neq f(S) \subsetneq S)$, see [11, Problem 4.12]. Sufficiently old papers refer to the Kinna–Wagner selection principle simply as the *selection principle*.

Thus, HT is simply the statement that every infinite set must be H-infinite. Hence, it follows from, e.g., [2, Proposition 4.2] that HT is not provable in ZF alone. We can get much more precise information after establishing the following equivalence of HT.

Proposition 2.2. *In* ZF*, the statement* HT *is equivalent to the statement that for every infinite set* X*, its finite powerset* $[X]^{<\omega}$ *is Dedekind-infinite.*

Proof. By [2, Theorem 3.2], a set X is H-finite if and only if $[X]^{<\omega}$ is Dedekind-finite. Thus the proposition follows immediately. □

Corollary 2.1. *In* ZF*,* Fin = D-Fin *implies* HT*, which in turn implies Form 82.*

Proof. Immediate from Proposition 2.2. □

In particular, by taking any model of ZF in which Fin = D-Fin holds but AC fails, we see that HT is strictly weaker than the full Axiom of Choice. The fact that the implication Fin = D-Fin ⇒ HT is not reversible is established in Sec. 3. That the implication HT ⇒ Form 82 is not reversible is the content of Sec. 4.

In light of Proposition 2.2, we see that HT is precisely the piece that is missing from either KL or RT to get Fin = D-Fin, as shown by the following proposition.

Proposition 2.3. *In* ZF*, the following are equivalent:*

(1) Fin = D-Fin*,*

(2) RT ∧ HT*,*

(3) KL ∧ HT*.*

Proof.

(1)⇒(2): This is immediate from Corollary 2.1 together with the well-known fact that Fin = D-Fin ⇒ RT.

(2)⇒(3): Immediate from the fact that RT ⇒ KL.

(3)⇒(1): Assume that HT and KL both hold, and let X be an arbitrary infinite set. By Proposition 2.2, HT implies that $[X]^{<\omega}$ is Dedekind-infinite and so there is a countable injective sequence $\langle F_n | n < \omega\rangle$ of finite subsets of X. Recursively replacing, if necessary, each F_n with $F_m \setminus \left(\bigcup_{k<m} F_k\right)$, where $m \geq n$ is the least index such that this set is nonempty, we may assume that the F_n are pairwise disjoint and nonempty. The sequence of F_n forms a countable family of nonempty finite sets, so by König's lemma there is a choice function $f : \omega \longrightarrow \bigcup_{n<\omega} F_n \subseteq X$. Since the F_n are pairwise disjoint and each $f(n) \in F_n$, we conclude that the function $f : \omega \longrightarrow X$ is in fact injective, and so X is Dedekind-infinite.

□

We finish the section with a couple more ZF results that will be useful in the next section. To state the first one, we recall a definition from [8].

Definition 2.2 ([8], Definition 8)**.** A set X is said to be *C-finite* if there is no surjection $f : X \longrightarrow \omega$, and it is *C-infinite* if it is not C-finite.

C-finite sets were called *dually Dedekind-finite* by Degen [4]. It follows from [7, Lemma 4.11] that any set X is C-finite if and only if $\wp(X)$ is Dedekind-finite. In particular, Form 82 can be thought of as the statement that every infinite set is C-infinite.

Recall also that a set is *amorphous* if it is infinite and its only subsets are the finite ones and the cofinite ones.

Proposition 2.4. *In* ZF*, if X is amorphous and C-infinite, then X is H-infinite.*

Proof. Since $\wp(X)$ is Dedekind-infinite, there is an injective sequence $\langle A_n \big| n < \omega\rangle$ of subsets of X. Now, since X is amorphous, each A_n is either a finite, or a cofinite, subset of X; using the pigeonhole principle, thin out the sequence by eliminating terms so that either all of the A_n are finite, or all of the A_n are cofinite. In the first case, let $F_n = A_n$; in the second case let $F_n = X \setminus A_n$, for all $n < \omega$. In either case, the sequence $\langle F_n \big| n < \omega\rangle$ is an injective sequence of elements of $[X]^{<\omega}$, and we are done. □

The next proposition, which is the last of the section, will be useful when determining whether HT holds in Cohen's model for the failure of the AC.

Proposition 2.5. *In* ZF*, if X is a linearly orderable H-infinite set, then X is Dedekind-infinite*

Proof. Let $\leq$ be a linear order on X and, since X is H-infinite, let $\langle F_n \big| n < \omega\rangle$ be an injective sequence of finite subsets of X. Using the same trick as in the proof of Theorem 2.3, we may assume that the F_n are pairwise disjoint. Hence, if we define $x_n = \min_{\leq} F_n$, the sequence $\langle x_n \big| n < \omega\rangle$ of elements of X is injective. Therefore, X is Dedekind-infinite. □

3. Models of ZF and ZFA

There are two main techniques for independence proofs that we use throughout this paper. The first one is by means of the forcing technique, passing to a special submodel of a forcing extension to get a model of ZF; models obtained in this way are called *symmetric models*. The only model arising from this technique that we will study in detail is Cohen's basic model, as described in [11, Sec. 5.3]; this model is denoted $\mathcal{M}_1$ in [10]. The other technique that will be used is that of the Fraenkel–Mostowski permutation models of ZFA, as described in [11, Secs. 4.1 and 4.2]. The three "classical" Fraenkel–Mostowski models that we will study in this section are the First and Second Fraenkel Model (denoted by $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, in [10]), and Mostowski's Linearly Ordered Model ($\mathcal{N}_3$ in [10]). These models are described (each on a different section) in [11, Secs. 4.3–4.5], and any unexplained notation is used as in that source. In Sec. 4, we will build a new permutation model in order to show that Form 82 does not imply HT.

3.1. **Transferable statements and finiteness classes.** Since we are ultimately interested in proofs of independence from ZF, rather than from ZFA, it is necessary to justify that independence proofs from the latter can be transferred to independence proofs from the former, for statements like the ones we will consider in this paper.

Definition 3.1. Let φ be a formula in the language of set theory.

(1) If φ is a statement, we say that φ is *transferable* if there is a metatheorem stating that, if there exists a Fraenkel–Mostowski model $\mathcal{N}$ of ZFA satisfying φ, then there exists a model $\mathcal{M}$ of ZF that also satisfies φ.

(2) We say that φ is a *boundable formula* if there is an absolutely definable ordinal α such that, for every x, we have that $\varphi(x)$ is equivalent to its relativization $\varphi^{\wp^\alpha(x)}(x)$ (here $\wp^\alpha$ denotes the usual iterated powerset operation, defined recursively by $\wp^0(x) = x$, $\wp^{\xi+1}(x) = \wp(\wp^\xi(x))$, and $\wp^\xi(x) = \bigcup_{\beta<\xi} \wp^\beta(x)$ for limit ξ).

(3) A *boundable statement* is the existential closure of a boundable formula.

(4) We say that φ is *injectively boundable* if it is a (finite) conjunction of formulas of the form

$$(\forall y)(\aleph(y) \leq \sigma(x) \Rightarrow \psi(y, x))$$

where $\psi(y, x)$ is a boundable formula and $\sigma(y)$ is a term[3] defined by a boundable formula that depends on y (here $\aleph(y)$ is the Hartogs number of y, the least ordinal number that does not inject in y).

(5) An *injectively boundable statement* is the existential closure of an injectively boundable formula.

The definitions of a boundable formula and statement are from [12], and all of the other definitions can be found in [13]. The classical Jech–Sochor theorem [12] (see also [11, Theorem 6.1]) states that all boundable statements are transferable. A generalization of this result was established by Pincus [13, Metatheorem 2A6], who proved that all injectively boundable statements are transferable (note that the class of injectively boundable statements contains all boundable statements and is closed under conjunction, so Pincus's result is stronger that Jech–Sochor's). An even stronger result that will be enough for our purposes is the following.

Theorem 3.1. *Any conjunction of a finite number of injectively boundable statements together with any statements among* OP*,* BPI*,* DC*,* CC*, is transferable.*

Proof. This is a consequence of [14, Theorem 4 and note in p. 145] (see also [15, p. 547]). □

Pincus's results are even more general (a much more general transfer theorem is stated in [10, p. 286]); here we have stated only what can be expressed in terms of the definitions given so far, which will be enough for our purposes.

Recall that a *finiteness class* is a class of sets $\mathscr{F}$ containing all finite sets, not containing ω, and closed under subsets and bijective images. It is worth noting that all the variations of "finite" that we have mentioned here (namely H-finite, Dedekind-finite and C-finite) constitute finiteness classes.

Definition 3.2. We will say that a finiteness class $\mathscr{F}$ is *tame* if there is a boundable formula $\varphi(x)$ such that $\mathscr{F} = \{x \mid \varphi(x)\}$.

A glance at the definitions will convince the reader that the classes of H-finite, Dedekind-finite and C-finite sets are all tame (this is also explained, with some more detail, in [2, p. 15, third paragraph]).

Theorem 3.2. *Let $\mathscr{F}, \mathscr{G}$ be tame finiteness classes. Then, both the statement that $\mathscr{F} = \mathscr{G}$ and the statement that $\mathscr{F} \neq \mathscr{G}$ are injectively boundable (and hence transferable).*

[3]For the purposes of this paper, it is always sufficient to take $\sigma(y) = \aleph_0$.

Proof. Let $\varphi(x), \psi(x)$ be boundable formulas such that $\mathscr{F} = \{x\big|\varphi(x)\}$ and $\mathscr{G} = \{x\big|\psi(x)\}$. Note that the class of boundable formulas is closed under Boolean combinations (conjunctions, disjunctions and negations), and that every boundable formula (respectively, statement) is also injectively boundable (respectively, statement). Hence, the statement $\mathscr{F} \neq \mathscr{G}$, which is equivalent to $(\exists x)((\varphi(x) \wedge \neg\psi(x)) \vee (\neg\varphi(x) \wedge \psi(x)))$, is boundable, hence injectively boundable.

For the remaining statement, recall that the class of Dedekind-finite sets (denoted D-Fin) is the largest finiteness class (this is a consequence of the fact that finiteness classes do not contain ω and are closed under subsets), and so $\mathscr{F}, \mathscr{G} \subseteq$ D-Fin. Hence, the statement that $\mathscr{F} = \mathscr{G}$ is equivalent to the statement that for every Dedekind-finite set x, $x \in \mathscr{F} \iff x \in \mathscr{G}$. It follows immediately from the definition that a set x is Dedekind-finite if and only if $\aleph(x) \leq \omega$. Hence, the statement that $\mathscr{F} = \mathscr{G}$ is equivalent to the statement

$$(\forall x)(\aleph(x) \leq \omega \Rightarrow (\varphi(x) \iff \psi(x))),$$

which is an injectively boundable statement. □

3.2. **The truth-value of HT in the models.** We now proceed to determine whether HT holds in each of the four models mentioned at the beginning of the section.

Theorem 3.3. HT *does not hold in the First Fraenkel Model* $\mathcal{N}_1$*.*

Proof. In $\mathcal{N}_1$ there is an infinite, H-finite set. In fact, the set A of atoms is such a set by [2, Proposition 4.2]. □

Theorem 3.4. *In the Second Fraenkel Model* $\mathcal{N}_2$*,* HT *holds.*

Proof. Take an arbitrary infinite set $X \in \mathcal{N}_2$ and let us argue that its finite powerset $[X]^{<\omega}$ is Dedekind infinite. If X is well-orderable we are done, so assume that it is not. Take a finite support $F_0 := \bigcup_{i=0}^{n_0} P_i$ for X. Now, working in the real world (rather than in $\mathcal{N}_2$), recursively choose $x_k \in X$ and $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$ and x_k is not supported by $F_k = \bigcup_{i=0}^{n_k} P_i$ but it is supported by $F_{k+1} = \bigcup_{i=0}^{n_{k+1}} P_i$ (we can always choose such an x_k because X fails to be well-orderable in $\mathcal{N}_2$ and so no single finite subset of A can simultaneously support every element of X). For each $k < \omega$, the set $Y_k = \{\pi(x_k)\big|\pi \text{ pointwise fixes } F_0\}$ is symmetric (supported by F_0) and hence it belongs to $\mathcal{N}_2$; as X is supported by F_0, we have $Y_k \subseteq X$. Furthermore, note that, since x_k is supported by F_{k+1}, the value of $\pi(x_k)$ is completely determined by $\pi \restriction F_{k+1}$, whenever $\pi \in G$. There are only finitely many possible values for $\pi \restriction F_{k+1}$ —in fact, with the requirement that π pointwise fixes F_0, and given that $F_{k+1} \setminus F_0 = \bigcup_{i=n_0+1}^{n_{k+1}} P_i$, there are at most $2^{n_{k+1}-n_0}$ possible values for $\pi \restriction F_{k+1}$. This means that $|Y_k| \leq 2^{n_{k+1}-n_0}$, so Y_k is finite. As each Y_k is supported by F_0, we may conclude that the set

$$f = \big\{\langle k, Y_k\rangle\big|k < \omega\big\}$$

is also supported by F_0, and hence $f \in \mathcal{N}_2$. While f need not be injective, notice that (since the x_k are pairwise distinct and each Y_k is finite) its range must be infinite, hence it is possible to modify f and obtain an injective function $: \omega \longrightarrow [X]^{<\omega}$. Thus, we may conclude that $[X]^{<\omega}$ is a Dedekind-infinite set in $\mathcal{N}_2$. Therefore $\mathcal{N}_2 \vDash$ HT. □

Theorem 3.5. *In Mostowski's Linearly Ordered Model* $\mathcal{N}_3$*,* HT *fails.*

Proof. It is known [11, Section 4.5] that $\mathcal{N}_3 \vDash$ OP, and this implies that $\mathcal{N}_3 \vDash$ KL; however, $\mathcal{N}_3 \nvDash$ Fin = D-Fin. Hence, by Proposition 2.3, it must be the case that $\mathcal{N}_3 \nvDash$ HT. □

Theorem 3.6. HT *fails in the Basic Cohen Model* $\mathcal{M}_1$*.*

Proof. Let X be the set of countably many generic Cohen reals used to construct $\mathcal{M}_1$. X is Dedekind-finite in $\mathcal{M}_1$ [3, Chap. IV, Sec. 9, Theorem 1, p. 138] (using modern notation, the argument in [11, Lemma 5.15] indeed shows this); it is also linearly orderable since $X \subseteq \mathbb{R}$. Hence, by Proposition 2.5, X is also H-finite and so $\mathcal{M}_1 \vDash \neg\mathsf{HT}$. □

3.3. **HT and BPI, OEP, KW, OP, KL, DC, CC, Fin = D-Fin.** With the information from the previous section under our belt, we are now able to establish which implications between HT and other choice principles can be proved under ZF. Recall that HT is simply the statement that the class of finite sets coincides with the class of H-finite sets; since both of these are tame finiteness classes, both HT and ¬HT are injectively boundable statements by Theorem 3.2. This fact will be used extensively in what follows.

Theorem 3.7. *Under* ZF*, the principles* DC*,* CC *and* Fin = D-Fin *imply* HT*, and none of these implications is reversible.*

Proof. Since DC ⇒ CC ⇒ Fin = D-Fin ⇒ HT (the latter implication due to Corollary 2.1), and both HT and ¬(Fin = D-Fin) are injectively boundable statements by Theorem 3.2, Theorem 3.1 implies that it suffices to exhibit a model of ZFA where HT holds but Fin = D-Fin fails. By Theorem 3.4, the Second Fraenkel Model $\mathcal{N}_2$ satisfies this (it is straightforward that the set of atoms A in $\mathcal{N}_2$ is Dedekind-finite). □

Theorem 3.8. *In* ZF*, there are no provable implications between* HT *and any of* BPI*,* OEP*,* KW*,* OP*,* KL *(thus* HT *is independent of each of these choice principles).*

Proof. Cohen's Basic Model $\mathcal{M}_1$ satisfies BPI and KW (this follows from [6], see also [10, p. 146]). Since BPI ⇒ OEP ⇒ OP ⇒ KL, and since HT fails in $\mathcal{M}_1$ (by Theorem 3.6), it follows that neither of KW, BPI, OEP, OP, or KL, imply HT in ZF.

(For all the choice principles mentioned in the statement of the theorem, except KW, one can obtain an alternative argument by considering Mostowski's Linearly Ordered Model $\mathcal{N}_3$, where HT fails (Theorem 3.5). This model satisfies BPI, as proved in [11, Section 7.1], so it suffices to invoke Theorems 3.5 and 3.1 to see that BPI does not imply HT over ZF (and hence, neither of OEP, OP, KL imply HT either). This argument, however, does not work for KW since the latter fails in $\mathcal{N}_3$, see [10, pp. 182–183].)

Conversely, consider the Second Fraenkel Model $\mathcal{N}_2$. By Theorem 3.4, $\mathcal{N}_2 \vDash \mathsf{HT}$. Since the set of atoms A in $\mathcal{N}_2$ is Dedekind-finite, it follows from Proposition 2.3 that $\mathcal{N}_2 \vDash \neg\mathsf{KL}$ (alternatively, one can directly see that the countable sequence of pairs that gives rise to the model, $\langle P_n \big| n < \omega\rangle$, constitutes a countable family of nonempty finite sets without a choice function). Note that the formula $\varphi(x)$ stating "$x = (T, \leq)$ is an infinite, finitely branching tree without infinite branches" is boundable (equivalent to its relativization to $\wp^{\omega+1}(x)$) and thus $\neg\mathsf{KL} \equiv (\exists x)(\varphi(x))$ is a boundable statement. Hence, by Theorem 3.1, HT does not imply KL in ZF. It follows immediately that HT does not imply neither of OP, OEP, BPI, or KW either. □

3.4. **HT and various flavours of RT.** So far in this paper, we have only considered the version of Ramsey's theorem dealing with partitions of pairs. Variants of this result in other dimensions have, however, also been considered.

Definition 3.3. Given an $n \in \mathbb{N} \setminus \{1\}$, the symbol RT^n will denote the statement that, for every infinite set X and every colouring $c : [X]^n \longrightarrow 2$, there exists an infinite $Y \subseteq X$ such that $[Y]^n$ is c-monochromatic.

It is possible to change the above definition to deal with any finite number of colours, but in any case we wind up with an equivalent statement. More precisely, if we let $\mathsf{RT}^n(k)$ (with $n, k \in \mathbb{N} \setminus \{1\}$) be the statement that for every infinite set X and every colouring $c : [X]^n \longrightarrow k$, there exists an infinite $Y \subseteq X$ such that $[Y]^n$ is c-monochromatic, then it is a theorem of Forster and Truss [5, Lemma 2.2] that, for each given n, all of the statements $\mathsf{RT}^n(k)$ as k varies are equivalent under ZF. The same two authors also establish [5, Theorem 2.3] that, if $n \leq m$, then $\mathsf{RT}^m \Rightarrow \mathsf{RT}^n$ in ZF. We still write RT without exponent to refer to RT^2, and remind the reader that this statement is Form 17 in [10]. The statement $(\forall n)(\mathsf{RT}^n)$, on the other hand, is referred to as Form 325 in [10].

In [2, Definition 2.1], a set X is defined to be R^n-infinite if for every $c : [X]^n \longrightarrow 2$ there exists an infinite $Y \subseteq X$ such that $[Y]^n$ is c-monochromatic (that is, if the n-dimensional Ramsey's theorem holds at X); and of course X is R^n-finite if it is not R^n-infinite. Hence, RT^n is simply the statement that every infinite set is R^n-infinite, and statements about the veracity or failure of the principles RT^n can be thought of as statements about certain finiteness classes being equal. Furthermore, the class of R^n-finite sets is tame for every $n \in \mathbb{N} \setminus \{1\}$, and therefore any of the statements RT^n, $\neg\mathsf{RT}^n$, and their combinations (in conjunction) with HT and $\neg\mathsf{HT}$ are injectively boundable by Theorem 3.2.

Theorem 3.9. *In* ZF*, there is no provable implication relation between* HT *and any of the* RT^n *(*$n \in \mathbb{N} \setminus \{1\}$*), nor between* HT *and Form 325.*

Proof. Consider the Second Fraenkel Model $\mathcal{N}_2$. Theorem 3.4 establishes that $\mathcal{N}_2 \vDash \mathsf{HT}$; on the other hand, it is shown in [2, Proposition 4.7] that the set of atoms in $\mathcal{N}_2$ is R^2-finite. In particular, RT fails and *a fortiori*, so do each of the RT^n ($n \geq 3$) as well as Form 325. Since HT and all of the $\neg\mathsf{RT}^n$ are injectively boundable, it follows from Theorem 3.1 that HT does not imply any of the Ramsey-theorem-related choice principles in ZF.

Conversely, consider the First Fraenkel Model $\mathcal{N}_1$. We know by Theorem 3.3 that HT fails in $\mathcal{N}_1$. On the other hand, it is established in [2, Proposition 4.1] that the set A of atoms is R^n-infinite for all $n \geq 2$; furthermore, by [1, Lemma, p. 389], every non-well-orderable set from $\mathcal{N}_1$ contains an infinite subset which is in bijection to a cofinite subset of A. Hence, every infinite set in $\mathcal{N}_1$ contains an infinite subset which is either in bijection with ω, or with a cofinite subset of A; since both ω and A are R^n-infinite, it follows that every infinite set is R^n-infinite in $\mathcal{N}_1$. Hence, $\mathcal{N}_1 \vDash (\forall n \in \mathbb{N} \setminus \{1\})(\mathsf{RT}^n)$; since $\neg\mathsf{HT}$ and all of the RT^n are injectively boundable, it follows that neither of the RT^n imply, not even jointly (i.e. as Form 325), the principle HT in ZF[4]. □

3.5. HT **and other choice principles.** We finish the chapter by briefly mentioning how one can obtain further information, regarding the implication relations (or lack thereof) between HT and a few other known choice principles. One can obtain plenty of information simply based on Theorems 3.4 and 3.6, which state that HT holds in the Second Fraenkel Model $\mathcal{N}_2$ and fails in Cohen's basic model $\mathcal{M}_1$. For example, there is no implication between HT and any of Choice from Well-Orderable sets $\mathsf{AC}(\infty, \mathrm{WO})$, Choice from finite sets $\mathsf{AC}(\infty, < \aleph_0)$, and Choice from pairs $\mathsf{AC}(\infty, \leq 2)$, since each of these principles holds in $\mathcal{M}_1$ (for the first one, see [11, Exercise 5.22]; the remaining two are easily consequences of OP) and fails in $\mathcal{N}_2$ (as witnessed by the partition of the set of atoms in pairs giving rise to the model); of course it is also important to notice that the failure of each of these principles is a boundable statement so we are able to use transfer theorems. It is

[4]It is clear that, for each individual n, the statement $\mathsf{RT}^n \wedge \neg\mathsf{HT}$ is injectively boundable. The statement of Form 325, however, is at first sight a conjunction of *all* of the RT^n simultaneously. In this case, one needs to verify by hand that the formula $\varphi(x)$ stating that "for every $n < \omega$ and every $c : [x]^n \longrightarrow 2$, there is an infinite $y \subseteq x$ such that c is constant in $[y]^n$" is boundable (equivalent to its relativization to $\wp^{\omega+1}(x)$) and hence Form 325, which is equivalent to $(\forall x)(\aleph(x) \leq \omega \Rightarrow \varphi(x))$, is an injectively boundable statement.

also worth noting that neither the Hahn–Banach theorem nor the Ultrafilter theorem imply HT, not even jointly, since both of these principles follow from BPI; and HT does not imply either of these principles either, as witnessed by Solovay's model (which satisfies DC, and therefore also HT).

4. HT vs. Form 82

Recall that we established in Corollary 2.1 that HT implies Form 82 under ZF. The purpose of this section is to prove that this implication is not reversible. Since Form 82 is equivalent to the statement that every C-finite set is finite, and the class of C-finite sets is tame, it follows from Theorems 3.2 and 3.1 that it suffices to build a Fraenkel–Mostowski permutation model of ZFA satisfying Form 82, but not satisfying HT.

For the construction, we begin with a model of ZF with $|A| = \mathfrak{c}$, and take a bijection : $\omega^\omega \longrightarrow A$, which we denote by $f \longmapsto a_f$. Recall that ω^ω is naturally endowed with a metric space structure, given by declaring $d(f,g)$ to be 0 if $f = g$ and $\dfrac{1}{1+\Delta(f,g)}$ otherwise, where $\Delta(f,g) = \min\{k < \omega \big| f(k) \neq g(k)\}$. For each $s \in \omega^{<\omega}$ we let $U_s = \{f \in \omega^\omega \big| f \restriction |s| = s\}$; this is an open ball in ω^ω with radius $\frac{1}{|s|}$, and the collection $\{U_s \big| s \in \omega^{<\omega}\}$ of all such balls forms a basis for the topology in ω^ω induced by the aforementioned metric.

We consider the group $\mathscr{G}$ consisting of all permutations of A that are induced by isometries of ω^ω; that is, $\pi \in \mathscr{G}$ if and only if there exists an isometry $\varphi : \omega^\omega \longrightarrow \omega^\omega$ such that $\pi(a_f) = a_{\varphi(f)}$, in which case we will denote $\pi = \pi_\varphi$. Note that every isometry must map each of the basic open sets U_s to some U_t satisfying $|s| = |t|$. Hence, any such isometry gives rise to, and is entirely determined by, an "assembly" of permutations $\langle \varphi_s : \omega \longrightarrow \omega \big| s \in \omega^{<\omega}\rangle$ such that, for each $s \in \omega^{<\omega}$, if $\varphi[U_s] = U_t$ then $\varphi[U_{s^\frown n}] = U_{t^\frown \varphi_s(n)}$ for all $n < \omega$.

We now proceed to define a filter on $\mathscr{G}$. For each $n < \omega$ and finite $F \subseteq \omega^\omega$, we define

$$G_{n,F} = \{\pi_\varphi \big| (\forall f \in F)(\varphi(f) = f) \text{ and } (\forall s \in \omega^n)(\varphi[U_s] = U_s)\}$$

In other words, $G_{n,F}$ consists of all π_φ where, if φ is determined by the assembly of permutations $\langle \varphi_s \big| s \in \omega^{<\omega}\rangle$, then we have for all $k \leq n$ and for all $s \in \omega^k$ that φ_s is the identity permutation, and furthermore, for all $k < \omega$ and all $f \in F$ it is the case that $\varphi_{f\restriction k}(f(k)) = f(k)$. (Note that $G_{0,\varnothing} = \mathscr{G}$.) It is easily verified that, for $n, m < \omega$ and finite $E, F \subseteq \omega^\omega$, we have $G_{n,F} \cap G_{m,E} = G_{\max\{n,m\},E\cup F}$, and so the family $\{G_{n,F} \big| n < \omega \text{ and } F \subseteq \omega^\omega \text{ is finite}\}$ generates a filter of subgroups of $\mathscr{G}$, which we will denote with $\mathscr{F}$. Furthermore, $\mathscr{F}$ contains the stabilizer of each atom (the stabilizer of a_f is the subgroup $G_{0,\{f\}}$), and is closed under conjugates (since given $\pi_\varphi \in \mathscr{G}$, we have $\pi_\varphi^{-1} G_{n,F} \pi_\varphi = G_{n,\varphi^{-1}[F]}$). The filter $\mathscr{F}$ is therefore a normal filter of subgroups of $\mathscr{G}$, as defined in [11, Chap. 4], and so the class $M(A, \mathscr{F}, \mathscr{G})$ of hereditarily symmetric (with respect to this filter and group) sets satisfies ZFA.

Lemma 4.1. *In $M(A, \mathscr{F}, \mathscr{G})$, the set A is not H-infinite.*

Proof. Suppose, on the contrary, that there exists within $M(A, \mathscr{F}, \mathscr{G})$ a countable injective sequence $\langle A_m \big| n < \omega\rangle$ with each A_m a finite subset of A. Let $n < \omega$ and $F \in [A]^{<\omega}$ be such that the enumeration of this sequence is fixed by the elements of $G_{n,F}$. Since the A_m are mutually distinct, there is a $k < \omega$ such that $A_k \not\subseteq F$, so we may pick an $f \in \omega^\omega$ such that $a_f \in A_k \setminus F$, pick $K > \max\{n\} \cup \{\Delta(f,g) \big| g \in F\}$, and let $f' \in U_{f\restriction K} \setminus \{g \in \omega^\omega \big| a_g \in A_k\}$ (this can be done because A_k is finite). Then one can find an isometry $\varphi : \omega^\omega \longrightarrow \omega^\omega$ fixing all U_s for $s \in \omega^n$, fixing each element of F, and mapping f to f'. Thus, the permutation $\pi_\varphi \in G_{n,F}$ maps a_f to $a_{f'}$ and consequently does not fix A_k, contradicting the fact that it fixes the sequence $\langle A_m \big| m < \omega\rangle$. □

In particular, we have that $M(A, \mathscr{F}, \mathscr{G}) \vDash \neg\mathsf{HT}$. The remainder of the section is devoted to proving that every infinite set in $M(A, \mathscr{F}, \mathscr{G})$ is C-infinite, and hence this model satisfies Form 82. We begin by considering subsets of A.

Lemma 4.2. *Working in $M(A, \mathscr{F}, \mathscr{G})$, let $X \subseteq A$ be infinite. Then X is C-infinite.*

Proof. Begin by noticing that, for each $s \in \omega^{<\omega}$, the set $A_s = \{a_f \big| f \in U_s\}$ belongs to $M(A, \mathscr{F}, \mathscr{G})$, since this set is fixed by all elements of $G_{|s|,\varnothing}$. Furthermore, each permutation in $G_{|s|+1,\varnothing}$ fixes each of the sets $A_{s^\frown n} = \{a_f \big| f \in U_{s^\frown n}\}$; this implies that the (injective) sequence $\langle A_{s^\frown n} \big| n < \omega\rangle$, consisting of subsets of A_s, also belongs to $M(A, \mathscr{F}, \mathscr{G})$. This shows that, in $M(A, \mathscr{F}, \mathscr{G})$, each of the sets A_s is C-infinite (in particular, for $s = \varnothing$ we see that $A = A_\varnothing$ is C-infinite). Thus, to prove the lemma it suffices to show that, if $X \subseteq A$ is infinite and belongs to $M(A, \mathscr{F}, \mathscr{G})$, then $A_s \subseteq X$ for some $s \in \omega^{<\omega}$.

To show this, suppose $X \subseteq A$ is infinite and belongs to $M(A, \mathscr{F}, \mathscr{G})$. Let $n < \omega$ and $F \in [A]^{<\omega}$ be such that each element of $G_{n,F}$ fixes X. Since X is infinite, we may find an $f \notin F$ such that $a_f \in X$. Pick an $m > \max\{\Delta(f, g) \big| g \in F\} \cup \{n\}$. Notice, then, that for each $f' \in U_{f\restriction m}$ it is possible to find an isometry $\varphi : \omega^\omega \longrightarrow \omega^\omega$ such that $\varphi[U_s] = U_s$ for $s \in \omega^m$, $\varphi(g) = g$ for $g \in F$, and $\varphi(f) = f'$. Since each such isometry φ satisfies $\pi_\varphi \in G_{n,F}$, we conclude that $a_{f'} = a_{\varphi(f)} = \pi_\varphi(a_f) \in X$. Therefore $A_{f\restriction m} \subseteq X$. □

We are now in conditions to finish our proof.

Theorem 4.1. *The model $M(A, \mathscr{F}, \mathscr{G})$ satisfies Form 82.*

Proof. Working in $M(A, \mathscr{F}, \mathscr{G})$, let X be an arbitrary infinite set. If X is well-orderable, then it is C-infinite, so we may assume without loss of generality that X is not well-orderable. This means that for no $n < \omega$ and for no $F \in [A]^{<\omega}$ can the permutations of $G_{n,F}$ simultaneously fix all elements of X. Let $n < \omega$ and $F \in [A]^{<\omega}$ be such that $G_{n,F}$ fixes X. We divide the proof in two cases:

Case 1: Suppose there is an $x \in X$ such that for no $m \geq n$ is it the case that $G_{m,F}$ fixes x. In this case, pick an $F' \in [A]^{<\omega} \setminus \{F\}$, with $F \subseteq F'$, of least possible cardinality such that $G_{m,F'}$ fixes x for some $m < \omega$. Fix such m, assuming without loss of generality that $m \geq n$. Choose $f \in F' \setminus F$, and choose $k > \max\big(\{m\} \cup \{\Delta(f, g) \big| g \in F'\}\big)$. Consider the set

$$h = \{(\pi(a_f), \pi(x)) \big| \pi \in G_{k,F'\setminus\{f\}}\}.$$

This set is clearly fixed by all permutations in $G_{k,F'\setminus\{f\}}$ and therefore belongs to $M(A, \mathscr{F}, \mathscr{G})$. Furthermore, we claim that h is a function. To see this, suppose that we have two elements $\pi_\varphi, \pi_\psi \in G_{k,F'\setminus\{f\}}$ such that $\pi_\varphi(a_f) = \pi_\psi(a_f)$. This means $\varphi(f) = \psi(f)$, and thus $\varphi^{-1}\psi(f) = f$. Hence, $\varphi^{-1}\psi \in G_{k,F'} \subseteq G_{m,F'}$ and so $\pi_\varphi^{-1}(\pi_\psi(x)) = \pi_{\varphi^{-1}\psi}(x) = x$, and therefore $\pi_\varphi(x) = \pi_\psi(x)$. Note also that, since $G_{k,F'\setminus\{f\}} \subseteq G_{n,F}$, the range of h is a subset of X.

Now, for each $j \geq k$, notice that the set $A_j = \{a_g \in A \big| \Delta(f, g) = j\}$ belongs to $M(A, \mathscr{F}, \mathscr{G})$ (as it is fixed by all elements in $G_{0,\{f\}}$), and let us consider the restricted function $h \restriction A_j$. The argument breaks into two subcases:

Subcase 1.A: There exists a $j \geq k$ such that $h \restriction A_j$ is injective. Then the set $h[A_j]$ is in bijection with A_j; since the latter is C-infinite by Lemma 4.2, we conclude that so is $h[A_j]$. Since $h[A_m] \subseteq X$, it follows that X is C-infinite.

Subcase 1.B: For every $j \geq k$, the function $h \restriction A_j$ fails to be injective. We claim that, in this case, each of the functions $h \restriction A_j$ is in fact a constant function, with constant value x. To see this fix any $j \geq k$, and pick two permutations $\pi, \rho \in G_{k,F'\setminus\{f\}}$ such that $\pi(a_f), \rho(a_f) \in A_j$, $\pi(a_f) \neq \rho(a_f)$, and $\pi(x) = \rho(x)$. Then $\rho^{-1}\pi(a_f) \in A_j \setminus \{a_f\}$ and $\rho^{-1}\pi(x) = x$. We conclude that there is a $\sigma \in G_{k,F'\setminus\{f\}}$ such that $\sigma(a_f) \in A_j \setminus \{a_f\}$ and $\sigma(x) = x$.

Now take an arbitrary $g \in \omega^\omega$ such that $\Delta(g, f) = j$ (that is to say, take an arbitrary $a_g \in A_j$). Then one can find a $\tau \in G_{k,F'}$ such that $\tau(\sigma(a_f)) = a_g$; thus we have $\tau(x) = x$. Notice, then, that $\tau\sigma \in G_{k,F'\setminus\{f\}}$, and so $h(\tau\sigma(a_f)) = \tau\sigma(x)$. But $\tau(\sigma(a_f)) = a_g$ and $\tau(\sigma(x)) = \tau(x) = x$, thus $h(a_g) = x$ and, a_g being arbitrary in A_j, we conclude that $h \restriction A_j$ is a constant function with constant value x.

Note that, given any $\pi \in G_{k,F'\setminus\{f\}}$, we must have, if $\pi(a_f) = a_g$, that $\Delta(f, g) \geq k$. In other words, $\pi(a_f) \in A_j$ for some $j \geq k$. Then, $x = h(\pi(a_f)) = \pi(x)$. The conclusion is that $G_{k,F'\setminus\{f\}}$ fixes x, contrary to the assumption about F' being of least possible cardinality.

Case 2: Suppose the assumption from Case 1 does not hold. Then, for every $x \in X$, there is an $m \geq n$ such that $G_{m,F}$ fixes x. Since $G_{n,F}$ fixes X setwise, we have an action of $G_{n,F}$ on the set X, and therefore X can be written as the disjoint union of orbits. Each of these orbits is of the form $\mathcal{O}(x) = \{\pi(x) \big| \pi \in G_{n,F}\}$, and is fixed by $G_{n,F}$ —hence it belongs to $M(A, \mathscr{F}, \mathscr{G})$. We begin by showing that every orbit is, in fact, well-orderable in $M(A, \mathscr{F}, \mathscr{G})$. To see this, take an arbitrary $x \in X$, and let $m \geq n$ be such that $G_{m,F}$ fixes x. Then we claim that $G_{m,F}$ fixes every element of $\mathcal{O}(x)$: for if $\pi(x) \in \mathcal{O}(x)$, for some $\pi \in G_{n,F}$, and $\rho \in G_{m,F}$ is arbitrary, then a routine calculation shows that $\pi^{-1}\rho\pi \in G_{m,F}$, and therefore $\pi^{-1}\rho\pi(x) = x$, thus $\rho(\pi(x)) = \pi(x)$. Since a single element of the filter $\mathscr{F}$ fixes all elements of $\mathcal{O}(x)$, a routine argument shows that any well-ordering of $\mathcal{O}(x)$ "from the real world" must also belong to $M(A, \mathscr{F}, \mathscr{G})$ —fixed by $G_{m,F}$.

In particular, this implies that X cannot be covered with finitely many orbits, since the disjoint union of finitely many well-orderable sets is well-orderable in ZF, contrary to our assumption about X. Now, the set $\mathscr{O} = \{\mathcal{O}(x) \big| x \in X\} \subseteq \wp(X)$ belongs to, and is well-orderable in, $M(A, \mathscr{F}, \mathscr{G})$ —it and each of its elements being fixed by $G_{n,F}$. In particular, it is Dedekind-infinite, and hence so is $\wp(X)$, which means that X is C-infinite, and we are done.

This finishes the proof. □

Corollary 4.1. *In* ZF*, Form 82 does not imply* HT*.*

Proof. By Lemma 4.1, the model $M(A, \mathscr{F}, \mathscr{G})$ fails to satisfy HT; this, coupled with Theorems 4.1, 3.2, and 3.1, finishes the proof. □

5. A weaker Boolean form of HT

In [2], certain variations of H-finite sets are considered. Recall that, given a set X, one can equip the set $[X]^{<\omega}$ with the symmetric difference as a binary operation, in order to obtain an Abelian group (in which each element has order 2, hence this is usually called a *Boolean group*). In this group, given a family $Y \subseteq X$ one can consider its set of *finite sums*, $\mathrm{FS}(Y) = \{y_1 + \ldots + y_m \big| m \in \mathbb{N} \wedge y_1, \ldots, y_m \in Y\}$, where $+$ denotes the Abelian group operation —the symmetric difference. Note that, for a pairwise disjoint family $Y \subseteq [X]^{<\omega}$, $\mathrm{FU}(Y) = \mathrm{FS}(Y)$ and so one could consider Hindman's finite unions theorem HT as a special case of an analogous statement on which one obtains a monochromatic

$\mathrm{FS}(Y)$ without requiring that Y is a pairwise disjoint family. It turns out that this analogous statement is equivalent to the original one; however, one obtains strictly weaker statements if one starts restricting the number of summands allowed in our finite sums.

Definition 5.1. Given a set X, a family $Y \subseteq [X]^{<\omega}$, and an $n \in \mathbb{N}$, we let $\mathrm{FS}_{\leq n}(Y) = \{F_1 \triangle \cdots \triangle F_t \big| t \leq n \text{ and } F_1, \ldots, F_t \in Y\}$. For $n, k \in \mathbb{N}$, we let $\mathsf{HT}_n(k)$ denote the statement that for every infinite set X and every colouring $c : [X]^{<\omega} \longrightarrow k$, there exists an infinite $Y \subseteq [X]^{<\omega}$ such that $\mathrm{FS}_{\leq n}(Y)$ is c-monochromatic.

Hence, $\mathsf{HT}_n(k)$ denotes Hindman's theorem for k colours and finite sums of at most n summands. For a fixed k, it follows[5] from [2, Theorem 3.2] that HT is equivalent to $\mathsf{HT}_n(k)$ whenever $n \geq 4$ and k is arbitrary, and $\mathsf{HT} \Rightarrow \mathsf{HT}_3(k) \Rightarrow \mathsf{HT}_2(k)$.

Looking at [2, Corollary 4.16] and the subsequent discussion, as well as [2, Question 5.1 (1)], it becomes apparent that $\mathsf{HT}_3(k)$ seems to be very close to HT, and for all we know these two principles could very well turn out to be equivalent over ZF.

Question 5.1. *Given a fixed $k \geq 2$, is the principle $\mathsf{HT}_3(k)$ equivalent to HT over ZF?*

In this section, we study the strength of the principle $\mathsf{HT}_2(k)$, where k is fixed but arbitrary. Although we do not know whether the $\mathsf{HT}_2(k)$ are equivalent for distinct k (of course, it is clear that $\mathsf{HT}_2(k+1) \Rightarrow \mathsf{HT}_2(k)$ for each $k \geq 2$), for the considerations in this paper it does not make a difference which specific k we have fixed, and so we will uniformly study all of the $\mathsf{HT}_2(k)$. Following [2, Definition 3.1, Corollary 3.5, Definition 3.6], we will say that X is $\mathrm{H}_2(k)$-infinite if for every $c : [X]^{<\omega} \longrightarrow k$ there exists an infinite $Y \subseteq [X]^{<\omega}$ such that $\mathrm{FS}_{\leq 2}(Y)$ is c-monochromatic, and X is $\mathrm{H}_2(k)$-finite if it is not $\mathrm{H}_2(k)$-infinite. So $\mathsf{HT}_2(k)$ is simply the statement that every infinite set is $\mathrm{H}_2(k)$-infinite, and hence both $\mathsf{HT}_2(k)$ and its negation are injectively boundable statements (since the class of $\mathrm{H}_2(k)$-finite sets is a tame finiteness class).

Since every R^2-infinite set is H_2-infinite by [2, Theorem 3.8], we have $\mathsf{RT} \Rightarrow \mathsf{HT}_2(k)$, for any $k \geq 2$. Hence, either of HT and RT (and *a fortiori* also any RT^n, $n > 2$, as well as Form 325) both imply $\mathsf{HT}_2(k)$ for every $k \geq 2$. We now proceed to precisely locate $\mathsf{HT}_2(k)$ among the various choice principles considered in this paper. The information obtained can be seen in the enhanced diagram from Fig. 2 (with the main information being the arrows *not* shown in the diagram, cf. the discussion in the Introduction).

Theorem 5.1. *The models $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ all satisfy $\mathsf{HT}_2(k)$ for all $k \geq 2$.*

Proof. Since $(\mathsf{HT} \vee \mathsf{RT}) \Rightarrow \mathsf{HT}_2(k)$, the statement is immediate from the fact that $\mathcal{N}_1 \vDash \mathsf{RT}$ (by [1, Theorem 2]), $\mathcal{N}_2 \vDash \mathsf{HT}$ (Theorem 3.4), and $\mathcal{N}_3 \vDash \mathsf{RT}$ (this is a theorem of Tachtsis [16, Theorem 2.4]). □

Corollary 5.1. *In ZF, $\mathsf{HT}_2(k)$ does not imply HT or any of the RT^n (and* a fortiori*, it does not imply Form 325 either), for any $k \geq 2$.*

Proof. Either of the models $\mathcal{N}_1$ and $\mathcal{N}_3$ satisfy $\mathsf{HT}_2(k)$ by Theorem 5.1, while neither of them satisfies HT, by Theorems 3.3 and 3.5. Thus $\mathsf{HT}_2(k)$ does not imply HT (in ZFA, but also in ZF since $\mathsf{HT}_2(k) \wedge \neg\mathsf{HT}$ is an injectively boundable statement), for any $k \geq 2$.

[5] In the two references that follow, what was really proved is the case $k = 2$, but it is clear from a cursory reading of the proof that this can be adapted to any k.

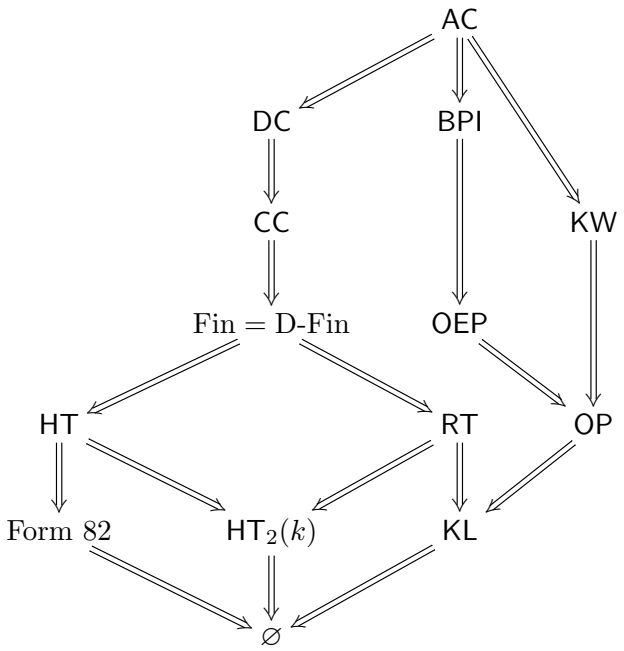

FIG. 2. Enhanced diagram of implication relations, now including $\mathsf{HT}_2(k)$ among the other classical choice principles.

On the other hand, the model $\mathcal{N}_2$ satisfies $\mathsf{HT}_2(k)$ but it does not satisfy any of the RT^n (as argued in the proof of Theorem 3.9), and so $\mathsf{HT}_2(k)$ does not imply any of the RT^n (again, originally in ZFA but it follows that this works also in ZF since $\mathsf{HT}_2(k) \wedge \neg\mathsf{RT}$ is an injectively boundable statement), for any $k \geq 2$. □

The choice principle $\mathsf{HT}_2(k)$ is thus strictly weaker than both HT, and all of the RT^n, for any $k \geq 2$. It follows immediately that $\mathsf{HT}_2(k)$ is also strictly weaker than (meaning that it is implied by, with the implication not reversible) all of DC, CC, and Fin = D-Fin. The following theorem will allow us to establish the lack of implication relations between $\mathsf{HT}_2(k)$ and a host of other choice principles.

Theorem 5.2. *In the basic Cohen model $\mathcal{M}_1$, $\mathsf{HT}_2(2)$ fails (and hence so does $\mathsf{HT}(k)$, for any $k \geq 2$).*

Proof. Let us fix some notation: let $\mathbb{P}$ be the forcing notion where conditions are finite functions $p : F \times n \longrightarrow 2$ for some finite $F \subseteq \omega$, $n < \omega$. Given a permutation $\pi \in \mathrm{Sym}(\omega)$, we also use the letter π to denote the automorphism $\pi : \mathbb{P} \longrightarrow \mathbb{P}$ induced by permuting the columns of $\omega \times \omega$ according to π. Let $X = \{x_n \big| n < \omega\}$ be the set of Cohen reals (thought of as elements of 2^ω) added by $\mathbb{P}$ (x_n represents the n-th column of the generic function $: \omega \times \omega \longrightarrow 2$). (Of course, the enumeration $\langle x_n \big| n < \omega\rangle$ does not belong to $\mathcal{M}_1$ even though the set $\{x_n \big| n < \omega\}$ does.)

Define the colouring $c : [X]^{<\omega} \longrightarrow 2$ given by $c(F) = 1$ if and only if there are distinct $x_n, x_m \in F$ such that $\min(x_n \triangle x_m)$ is even. We claim that c witnesses that $\mathsf{HT}_2(2)$ fails. To see this, suppose, aiming for a contradiction, that $Y \subseteq [X]^{<\omega}$ is an infinite set such that $\mathrm{FS}_{\leq 2}(Y)$ is c-monochromatic. Let $E \subseteq \omega$ be a finite "support" for the set Y; in other words, let $\mathring{Y}$ be a $\mathbb{P}$-name for Y such that every permutation π fixing each element of E, satisfies that $\pi(\mathring{Y}) = \mathring{Y}$. Since Y is infinite, there is an $F \in Y$ such that $F \not\subseteq \{x_k \big| k \in E\}$. Let $n < \omega$, $n \notin E$, and $p \in \mathbb{P}$ be such that

$p \Vdash$ "$\mathring{x_n} \in \mathring{F} \in \mathring{Y}$". Take an $m < \omega$ such that $m \notin \mathrm{dom}(\mathrm{dom}(p)) \cup E$, and let π be the transposition of n and m. Then $\pi(p) \Vdash$ "$\mathring{x_m} \in \pi(\mathring{F}) \in \mathring{Y}$"; furthermore, p and $\pi(p)$ are compatible conditions. Thus $p \cup \pi(p) \Vdash$ "$\{\mathring{x_n}, \mathring{x_m}\} = \mathring{F} \triangle \pi(\mathring{F}) \in \mathrm{FS}_{\leq 2}(\mathring{Y})$", and $p \cup \pi(p)$ also forces that x_n agrees with x_m up to $\mathrm{ran}(\mathrm{dom}(p))$. Let q be an extension of $p \cup \pi(p)$ deciding that $\min(\mathring{x_n} \triangle \mathring{x_m})$ is even; this shows that the colour of $\mathrm{FS}_{\leq 2}(Y)$ is 1. However, one can run the exact same argument up to the moment where one chooses q, at which time let us pick q forcing that $\min(\mathring{x_n} \triangle \mathring{x_m})$ is odd. This implies that the colour of $\mathrm{FS}_{\leq 2}(Y)$ is also 0, a contradiction. □

Corollary 5.2. *Given a fixed $k \geq 2$, there is no* ZF*-provable implication relation between* $\mathsf{HT}_2(k)$ *and neither of* BPI*,* OEP*,* KW*,* OP*, or* KL*.*

Proof. $\mathsf{HT}_2(k)$ cannot imply neither of the choice principles mentioned in the statement of the theorem, since HT does not imply them either. On the other hand, all of these choice principles hold in $\mathcal{M}_1$ (as mentioned in the proof of Theorem 3.8), while $\mathsf{HT}_2(k)$ fails in this model because of Theorem 5.2. Hence, neither of the principles mentioned in the statement of the theorem implies $\mathsf{HT}_2(k)$. □

It remains to establish whether there are any implication relations between $\mathsf{HT}_2(k)$ and Form 82. In order to do this, the following theorem will be instrumental.

Theorem 5.3. *In the model $M(A, \mathscr{F}, \mathscr{G})$ from Sec. 4,* $\mathsf{HT}_2(2)$ *fails (and hence so does each of the* $\mathsf{HT}_2(k)$ *for $k > 2$).*

Proof. Notice that the metric on A induced by the bijection $f \longrightarrow a_f$ from ω^ω to A belongs to $M(A, \mathscr{F}, \mathscr{G})$, since it is supported by the empty set; use the letter d to denote that metric. Then, for any two atoms a_f, a_g, the number $d(a_f, a_g)$ is the reciprocal of an integer. We can therefore define, within $M(A, \mathscr{F}, \mathscr{G})$, the colouring $c : [A]^{<\omega} \longrightarrow 2$ given by $c(F) = 1$ if and only if there are two distinct $a_f, a_g \in F$ such that $1/d(a_f, a_g)$ is even. We claim that the colouring c witnesses the failure of $\mathsf{HT}_2(2)$ in $M(A, \mathscr{F}, \mathscr{G})$.

To see this, suppose that $Y \subseteq [X]^{<\omega}$ is an infinite, pairwise disjoint family such that $\mathrm{FS}_{\leq 2}(Y)$ is monochromatic, and let $E \subseteq \omega^\omega$ be a finite set, and $n < \omega$, such that every $\pi \in G_{n,E}$ fixes Y. Since Y is infinite, we can find an $F \in Y$ such that there is $a_f \in F \setminus \{a_h \big| h \in E\}$. Find a $k > \max\{n\} \cup \{\Delta(g,h) \big| g, h \in E \cup \{f\} \cup \{h \big| a_h \in F\}\}$ and a $g \in \omega^\omega \setminus E$ such that $g \restriction k = f \restriction k$, $g \notin E$, $a_g \notin F$, and f differs from g for the first time at the odd number $m > k$. Letting π be induced by an isometry of ω^ω in such a way that π fixes all elements of $(F \setminus \{a_f\}) \cup \{a_h \big| h \in E\}$ and $\pi(a_f) = a_g$, we obtain that $\pi(F) = F \cup \{a_g\} \setminus \{a_f\} \in Y$ and therefore $\{a_f, a_g\} = F \triangle \pi(F) \in \mathrm{FS}_{\leq 2}(Y)$, with $1/d(a_f, a_g) = m + 1$ an even number. This shows that the color in which $\mathrm{FS}_{\leq 2}(Y)$ is monochromatic must be 1; however, running the exact same argument but choosing g in such a way that $m = \Delta(f, g)$ is even shows that this colour must be 0 as well, a contradiction. □

Corollary 5.3. *In* ZF*, there is no provable implication between* $\mathsf{HT}_2(k)$ *and Form 82, for any $k \geq 2$.*

Proof. By Theorem 5.3, the model constructed in Sec. 4 satisfies Form 82 together with $\neg\mathsf{HT}_2(k)$. So Form 82 does not imply $\mathsf{HT}_2(k)$, for any $k \geq 2$ (originally in ZFA, but these are all statements concerning tame finiteness classes and so they are injectively boundable by Theorem 3.2).

Conversely, consider the First Fraenkel Model $\mathcal{N}_1$. We know that $\mathcal{N}_1 \vDash \mathsf{HT}_2(k)$ (cf. Theorem 5.1). We claim that Form 82 fails in $\mathcal{N}_1$. To see this, note that the set of atoms A in $\mathcal{N}_1$ is amorphous (this is a classical and easy-to-see fact), while at the same time H-finite (by [2, Proposition 4.2]). Hence, A must be infinite C-finite, by Proposition 2.4. In particular, Form 82 fails in $\mathcal{N}_1$ and so

$\mathsf{HT}_2(k)$ does not imply Form 82, for any $k \geq 2$ (once again, in ZFA but the statement is transferable to ZF by Theorem 3.2). □

We note the curious fact that, in each of the models considered in this section (or in the paper), the principles $\mathsf{HT}_2(k)$ either fail for all $k \geq 2$, or hold for all $k \geq 2$. The argument from Proposition 2.1, however, does not seem to translate well when the finite sums allowed are limited to a bounded number of summands. Hence, we close the paper with the following natural problem.

Question 5.2. *Are all the statements* $\mathsf{HT}_2(k)$*, for varying* $k \geq 2$*, equivalent over* ZF*? Does there exist a* ZF *model satisfying, e.g.,* $\mathsf{HT}_2(2)$ *but not* $\mathsf{HT}_2(3)$*?*

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