Universidad Nacional Autónoma de
México y Universidad Michoacana
de San Nicolás de Hidalgo

Posgrado Conjunto en Ciencias Matemáticas
UMSNH-UNAM

FINITENESS CLASSES FROM PARTITIONS

T E S I N A

que para obtener el grado de

Maestro en Ciencias Matemáticas

presenta

Angel Jareb Navarro Castillo
jareb@matmor.unam.mx

Asesor:
Doctor en Ciencias
Osvaldo Guzmán González
oguzman@matmor.unam.mx

Morelia, Michoacán, México
28 de junio de 2024

Contents

FINITENESS CLASSES FROM PARTITIONS

Angel Jareb Navarro Castillo

Resumen

En este trabajo estudiamos clases de conjuntos con particiones sin funciones de elección parcial infinita, tal que estas particiones y sus elementos cumplan criterios específicos. En particular, para cada cardinal κ, consideramos la clase de aquellos conjuntos con una partición de tamaño κ en conjuntos finitos, sin funciones de elección parcial infinitas. Estudiamos las relaciones entre estas clases con diferentes valores de κ. También probamos que ningún conjunto puede tener dos particiones en conjuntos finitos, tal que dichas particiones sean bien ordenables y tengan tamaños distintos. Sin embargo, encontramos que es consistente que si haya este tipo de particiones con distintos tamaños cuando las particiones en cuestión no son bien ordenables. Probamos enunciados en ZF y hacemos pruebas de consistencia mediante el uso de modelos de permutaciones.

Abstract

In this work, we study classes of sets with partitions without infinite partial choice functions, such that these partitions and their elements satisfies specific conditions. In particular, for every cardinal κ, we consider the class of sets with some partition of size κ consisting of finite sets, such that this partition does not have any infinite partial choice function. We study the relation between these classes with different instances of κ. We also prove that, for every set, it can not have two partitions, consisting of finite sets, such that these partition are well ordered and have different sizes. But, we find that it is consistent that there exists this kind of partitions of different sizes, when these are not well ordered. We prove statements in ZF, and carry out consistency proofs using permutation models.

Introduction

Since it was stated, the Axiom of Choice has been object of controversy, because it allows the existence of mathematical objects without any kind of description or construction. For example, the Axiom of Choice implies the existence of a well order for the set of real numbers.

The Axiom of Choice has very important consequences in many areas of mathematics. The logical relations between diverse consequences of the Axiom of Choice is an interesting topic in set theory. Equivalently, the logical relations between the negations of these consequences.

An important consequence of the Axiom of Choice is that every set is finite if and only if it is Dedekind finite (definition 1.3.2). Without the Axiom of Choice, we could have infinite sets that are Dedekind finite. But, in ZF, every finite set is Dedekind finite. Then, the class of finite sets is a subclass of the class of Dedekind finite sets.

A finiteness class (definition 1.3.3) is a class of sets that is an intermediate class between the class of finite sets and the class of Dedekind finite sets.

Studying the relations (of subclass) between finiteness classes is another approach to understanding logical relations between consequences of the axiom of choice.

In this work, we study finiteness classes defined as classes of sets that has partitions without infinite partial choice functions, such that the partition satisfies specific properties.

In chapter 1, we state basic set theoretical results. In chapter 2, we explore a general notion of classes of sets with partition without infinite partial choice functions; we state results about the specific classes of sets with partitions of size κ consisting of finite sets, with κ an infinite cardinal number; and we prove that there are no sets with partitions of different size consisting of finite sets. In chapter 3, we carry out the consistency proofs that complete the study started in the chapter 2.

Chapter 1

Preliminaries

The results in this chapter are either folklore or attributed to other authors. In the rest of the work (except for section 3.1), the results not cited are our own.

1.1 Language and Axioms of Set Theory

For a formal presentation and study of the axioms and the language of set theory, refer to chapter 3 of [3]. In this section we will present an informal resume.

A formula ϕ in the language of set theory is a sequence of symbols including $\in, (,), \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$ and a countable list of variables $x_0, x_1, \ldots$ that adhere to certain grammatical rules, providing it with meaning. Typically, we will use other letters or symbols for the variables. We will refer to ϕ simply as a formula.

We say that a variable x is free in ϕ (or just free variable, if there is no confusion) if it appears in ϕ without the influence of the quantifiers $\forall$ and $\exists$.

If in ϕ appears at most the free variables $y_0, \ldots, y_n$, we will also denote ϕ as $\phi(y_0 \ldots y_n)$. For sets $a_0, \ldots, a_n$, we write $\phi(a_0, \ldots, a_n)$ if the sets a_i satisfy the formula ϕ.

The set theory is all about sets, but we will use the notion of classes. A class $\mathcal{C}$ is defined by a formula ϕ, and we say that a set a belongs to $\mathcal{C}$ ($a \in \mathcal{C}$) if and only if $\phi(a)$.

The axioms of Zermelo-Fraenkel are the standard axioms of set theory (see [9], chapter 2). The list is denoted by ZFC. If we do not consider the Axiom of Choice, we denote it as ZF. We also use ZF$^-$, which is ZF without the Axiom of Foundation. For any collection of axioms T, and a statement ϕ, we denote by $T + \phi$ the collections of statements in T along with ϕ.

We will use another list of axioms: ZFA, wich corresponds to set theory with atoms. It will be explained in chapter 3.

Our base theory is ZF$^-$. Applications of the Axiom of Choice or the Axiom of Foundation will be stated explicitly.

1.2 Basic Definitions and Notation

For every set X, we denote $P(X) = \{A : A \subseteq X\}$.

For every function f, and X a subset of its domain, we denote

$$f[X] := \{f(x) : \ x \in X\}.$$

For a function f and sets X and Y, we denote $f; X \to Y$ if the domain of f is a subset of X and its image is contained in Y.

For any sets X and Y, we write $|X| = |Y|$ if there exists a bijective function $f : X \to Y$. In this case, we say that X is equipotent to Y. We write $|X| \leqslant |Y|$ if there is an injective function from X to Y, and $|X| < |Y|$ if $|X| \leqslant |Y|$ but $|X| \neq |Y|$.

The following is a classical result in Set Theory. The reader may refer to it as Theorem 3.17 in [3].

Theorem 1.2.1 (Cantor-Bernstein Theorem)**.** *For every sets X and Y, if $|X| \leqslant |Y|$ and $|Y| \leqslant |X|$, then $|X| = |Y|$.*

We denote by ω the set of natural numbers, and say that a set X is countable if $|X| \leqslant \omega$.

We say that a set X is finite if there exists a natural number n such that $|X| = |n|$. Conversely, a set is infinite if it is not finite.

Definition 1.2.2. For any set X, we denote the group

$$S_X = \{f : \ f : X \to X \text{ is biyective}\}$$

and we refer to this as the *symmetric group of X*.

1.3 The Axiom of Choice and Related Propositions

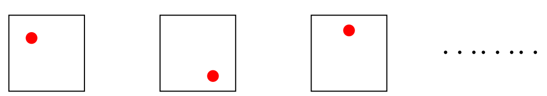

Figure 1.1: Representation of the Axiom of Choice

Definition 1.3.1. If $\mathscr{F}$ is a family of sets, we say that a function $f : \mathscr{F} \to \bigcup \mathscr{F}$ is a choice function of $\mathscr{F}$ if for every $X \in \mathscr{F}$ we have that $f(X) \in X$. We say that a function g is an infinite partial choice function of $\mathscr{F}$ if there exists $\mathscr{F}' \subseteq \mathscr{F}$, such that $\mathscr{F}'$ is infinite, and g is a choice function of $\mathscr{F}'$.

The Axiom of Choice (AC) states that, for every family of nonempty sets $\mathcal{F}$, there exists a choice function. We will consider the following statements that are equivalent to AC:

1. For every set X and P partition of X, there exists a choice function.

2. For every set X there is a well-order relation $\leqslant$ on X. This means that for every $A \subseteq X$, A has a minimum element relative to $\leqslant$.

Since cardinal numbers are ordinals, they are well orderable. For every well orderable set X, there exists a cardinal κ such that $|X| = |\kappa|$. Therefore, in the presence of the Axiom of Choice, every set is equipotent to some cardinal. Thus, in ZFC, it is meaningful to define $|X| := \kappa$, for every set X and cardinal κ such that X is equipotent to κ.

Definition 1.3.2. We say that a set X is Dedekind finite (or D-finite) if there is no injective function $f : X \to X$ such that $f[X] \subsetneq X$. In other words, X is not equipotent to a proper subset of it.

Observe that every finite set is Dedekind finite.

The following definition of finiteness class is due to H. Herrlich [4], but we can find an equivalent notion from J. W. Degen [1].

Definition 1.3.3. We say that a class $\mathcal{C}$ is a finiteness class if for every $X \in \mathcal{C}$: for every Y, if $|Y| \leqslant |X|$, then $Y \in \mathcal{C}$; every finite set belongs to $\mathcal{C}$; and $\omega \notin \mathcal{C}$.

Next, we will present two equivalent weakenings of the axiom of choice, which will be important in this work.

Definition 1.3.4.

1. The König's Lemma states that every tree of countable height with finite ramification has a cofinal branch.
2. The Axiom of Choice for countable families of finite sets (denoted as $\mathrm{C}(\omega, < \omega)$) states that every countable family consisting of non empty finite elements has a choice function.

Note that $\mathrm{C}(\omega, < \omega)$ is equivalent to state that every countable family consisting of non empty pairwise disjoint finite elements has a choice function.

Proposition 1.3.5 (Forster, Truss [2])**.** *The König's Lemma is equivalent to $C(\omega, < \omega)$.*

Proof. First we will assume $\mathrm{C}(\omega, < \omega)$. Let T be a tree of countable height with finite levels. For every $a \in T$ we will denote $S_a = \{b \in T : \ a \leqslant b\}$.

For every $s \in T$, denote

$$\mathrm{isucc}(s) = \{t \in T : \ t \text{ is succesor of } s \text{ and } S_t \text{ is infinite}\}.$$

Since T is countable, then $A = \{\mathrm{isucc}(s) : \ s \in T \text{ and } \mathrm{isucc}(s) \neq \emptyset\}$ is a family of countable sets. Then, there exists a choice function $g : A \to T$.

We will construct a cofinal branch for T by recursion. Consider $a_0 \in T$ as the root of the tree. See that the set S_{a_0} is infinite. Assume that we have $\{a_0, \ldots, a_n\}$ such that $\{a_0, \ldots, a_{n-1}\}$ is the set of predecessors of a_n and S_{a_n} is infinite. Since S_{a_n} is infinite, then $\mathrm{isucc}(a_n) \in A$. We select $a_{n+1} = g(isucc(a_n))$.

We conclude that $\{a_n : \ n \in \omega\}$ is a cofinal branch of T.

Now, we will assume the König's Lemma. Let $\{A_n : \ n \in \omega\}$ a family of non empty finite pairwise disjoint elements. We will consider the countable tree

$$T = \left\{ f : \{0, \ldots, m\} \to \bigcup_{n<m} A_n : \ m \in \omega \wedge (\forall i < m)(f(i) \in A_i) \right\}$$

ordered by the subset relation.

The root of T is $f_0 = \emptyset$, and it has countable many levels, where the n-th level consist of the elements of T with domain equal to $\{0, \dots, n-1\}$. Let $\{f_n : n \in \omega\}$ be a cofinal branch, where every f_n is the element of the branch in the level n of T. Then $f = \bigcup_{n\in\omega} f_n$ is a function from ω to $\bigcup_{n\in\omega} A_n$, and by construction this function is a choice function for $\{A_n : n \in \omega\}$. ■

In this work we will study generalizations of $\mathrm{C}(\omega, < \omega)$ (and therefore, the König's Lemma).

Definition 1.3.6. Let κ be an infinite cardinal. The Axiom of Choice for κ-families of non empty finite sets ($\mathrm{C}(\kappa, < \omega)$) states that every family of size κ consisting of finite elements has a choice function.

The following proposition will be useful later.

Proposition 1.3.7. *For every infinite well orderable set X, and P partition of X consisting of finite sets, we have $|P| = |X|$.*

Proof. First fix a well order $\leqslant$ for X. We will use the Cantor-Bernstein Theorem (theorem 1.2.1).

We have that $|P| \leqslant |X|$ due to the mapping that sends each element of P to its minimum.

Since $|P| \leqslant |X|$, P is well-orderable. Also, P is a partition of an infinite set consisting of finite sets, then P is infinite. Therefore, $|\omega| \leqslant |P|$.

Now, we can consider without loss of generality that $|P| = |\kappa|$ for some cardinal κ. We use the corollary 5.8 from [3], that give us that $|\kappa^{<\omega}| = |\kappa|$. Then we have that $|P^{<\omega}| = |P|$. Then,

$$|P \times \omega| \leqslant |P \times P| \leqslant |P^{<\omega}| = |P|.$$

But $|X| \leqslant |P \times \omega|$ due to the mapping that sends every $x \in X$ to the pair $(A, n) \in P \times \omega$, where $x \in A$ and n is the position of x in A according to the well order of X. Therefore, $|X| \leqslant |P|$.

■

The Axiom of Choice give us that every set is well orderable; consequently, here we have the following corollary.

Corollary 1.3.8. *Assuming the Axiom of Choice, for every infinite set X, and P partition of X consisting of finite sets, we have $|P| = |X|$.*

1.4 Consistency Proofs

We say that a collection of statements is consistent if we cannot deduce contradictions from it. We are interested in proving the consistency of $\mathcal{T} + \phi$ where ϕ is some statement and $\mathcal{T}$ is ZF or ZFA. It is the same as not being able to deduce $\neg\phi$ from $\mathcal{T}$.

Consider a formula ϕ, and a set M. We define ϕ^M as the formula that results from changing every occurrence of $\forall x$ and $\exists x$ to $\forall x \in M$ and $\exists x \in M$. This means that the variables are bounded by M.

Let ϕ be a statement. We say that a set M models ϕ if the statement ϕ^M holds. For this we write $M \models \phi$. If M models every element of a collection of statements T, we write $M \models T$.

By the Soundness Theorem (see theorem 3.3 in [3]), the Gödel Completeness Theorem (theorem 3.4 from [3]) and the Mostowski collapse lemma (see lemma I.9.35 in [7]), for a statement ϕ, we have that ZF+ϕ is consistent if there is some transitive set M (it means that, if $x \in M$, then $x \subseteq M$) such that $M \models \mathrm{ZF} + \phi$. Then, we are interested in finding such sets.

However, there is a problem. By Gödel's Second Incompleteness Theorem (theorem 3.9 from [3]), we can not prove the consistence of $\mathcal{T}$. Then, we work assuming the consistence of $\mathcal{T}$, and therefore the existence of a set V that models $\mathcal{T}$.

Refer to [8] for more information about consistency proofs in Set Theory.

In the chapter 3, we will define models of ZFA called Permutation Models. A permutation model will be consider as a class and not as a set. This is just for simplicity; actually, we can consider the permutation model inside of V, where V is the set that we obtain assuming the consistency of ZFA.

Chapter 2

Finiteness From Partitions

2.1 Partition Classes

We will examine finiteness classes defined by the existence of a specific partition. First we are going to study more general classes of sets.

Definition 2.1.1. Let $\phi(y, p_1, \ldots, p_n)$ and $\psi(x, y, t_1, \ldots, t_m)$ be formulas, $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ fixed sets. We say that a set X is $F(\phi_{a_1,\ldots,a_n}, \psi_{b_1,\ldots,b_m})$ if there exists a partition P of X without infinite partial choice functions such that both $\phi(P, a_1, \ldots, a_n)$ and $\psi(p, P, b_1, \ldots, b_m)$ hold for every $p \in P$. In this case we say that P witnesses that X is $F(\phi_{a_1,\ldots,a_n}, \psi_{b_1,\ldots,b_m})$.

The notation $F(\phi_{a_1,\ldots,a_n}, \psi_{b_1,\ldots,b_m})$ can be cumbersome; hence, occasionally, we will use the following alternatives.

- If there is no confusion, we will write ϕ instead of $\phi_{a_1,\ldots,a_n}$,
- if ϕ is equivalent to $0 = 0$ (in other words, ϕ is always true), we will denote ϕ as ∞,
- if A and B are sets: $F(A, \psi)$, $F(\phi, B)$, $F(< A, \psi)$ and $F(\phi, < B)$ denotes $F(|P| = |A|, \psi)$, $F(\phi, |p| = |B|)$, $F(|P| < |A|, \psi)$ and $F(\phi, |p| < |B|)$ respectively.
- if G and H denote the classes corresponding to the formulas ϕ and ψ, we will also denote $F(\phi, \psi)$ as $F(G, H)$.

We also use $F(\phi, \psi)$ to denote the class of sets that are $F(\phi, \psi)$.

For example, $F(\infty, \infty)$ is the class of the sets that have some partition without infinite partial choice functions, $F(\infty, < \omega)$ is the class of the sets that have some partition of finite sets without infinite partial choice functions, $F(\omega, < \omega)$ is the class of sets that have some countable partition of finite sets without infinite partial choice functions, and if $\mathcal{C}$ is a finiteness class, $F(\omega, \mathcal{C})$ is the class of sets that have some countable partition of sets in $\mathcal{C}$ without infinite partial choice functions.

We will particularly focus on the classes $F(\omega, < \omega)$ and $F(\infty, < \omega)$, which we will denote with simpler notations.

Definition 2.1.2. We define $F(\omega, <\omega)$ as the class of K-finite sets, with its elements referred to as K-finite sets. Similarly, $F(\infty, <\omega)$ represents the class of S-finite sets, and its elements are denoted as S-finite sets.

2.2 Some general properties about the classes $F(\phi, \psi)$

If ϕ is a formula with $n+1$ free variables and $a_1, \ldots, a_n$ are sets, we denote C_ϕ as the class of sets x which satisfies $\phi(x, a_1, \ldots, a_n)$.

We are interested in founding out which conditions of ϕ and ψ makes $F(\phi, \psi)$ a finiteness class. For this purpose, we state the next proposition.

Proposition 2.2.1. *Let ϕ and ψ be formulas. If C_ϕ contains every finite set, is closed under subsets and images of bijective functions, and C_ψ is a finiteness class, then $F(\phi, \psi)$ is a finiteness class.*

Proof. Let X and Y be sets such that $X \in F(\phi, \psi)$. Let P be the partition witnessing that $X \in F(\phi, \psi)$.

If $Y \subseteq X$, then $P' = \{Y \cap p : p \in P\} \backslash \{\emptyset\}$ is a partition of Y. Every $p' = Y \cap p$ in P' is a subset of the set p in X, and p satisfies ψ. As C_ψ is a finite class, then $p' \in C_\psi$. Furthermore, there exists an injective function $j : P' \to P$ where $j(p') = p$ for every $p' \in P'$. Since P satisfies ϕ, then P' also satisfies it. Finally, P' is a partition of Y that does not have an infinite partial function, as an infinite partial function of P' would induce one in P.

If $Y \sim X$, then the partition P induces a respective partition P' by a bijective function $f : X \to Y$. Then $P \sim P'$, and so P' satisfies ϕ. Additionally, for every $p \in P$, we have $p \sim f[p]$, and so $f[p]$ satisfies ψ. An infinite partial function in P' induces some in P by f. Therefore, P' does not have one. Consequently, $Y \in F(\phi, \psi)$.

If Y is a finite set, then $P' = \{\{y\} : y \in Y\}$ is a finite partition of Y that clearly does not have any infinite partial function. As P' is finite then satisfies ϕ, and every element in P' satisfies ψ since each of them is finite.

Every partition P of ω falls into one of two cases: it is either infinite or finite. If P is infinite, it has an infinite partial choice function induced by the well-ordering of ω. If P is finite, then at least one element A in P is infinite, then $A \sim \omega$ and therefore A does not satisfy ψ. Consequently, no partition of ω allows $\omega \in F(\phi, \psi)$. ■

Corollary 2.2.2. *The classes of S-finite and K-finite sets are a finite classes.*

Proposition 2.2.3. *Let A and B be sets, and $\mathscr{C}$ a class. If $\mathscr{C}$ is equal to one of the classes $F(A, B), F(<A, B), F(A, <B)$ or $F(<A, <B)$, then for all X, Y and Z such that $X \in \mathscr{C}$, $Y \subseteq X$ and $|Z| = |X|$, we have that $Y, Z \in \mathscr{C}$.*

Proof. We are going to consider $\mathscr{C} = F(A, B)$ and take $X \in \mathscr{C}$ such that $P_X = \{X_i : i \in I\}$ is its respective partition.

For $Y \subseteq X$, we have $P_Y = \{X_i \cap Y \mid i \in I\} \backslash \{\emptyset\}$ as a partition of Y that lacks any infinite partial choice function, because a function similar to one in P_Y would induce one in P_X. Since $\mathscr{C} = F(A, B)$, we have $|P_Y| \leqslant |P_X| \leqslant |A|$, and for all $i \in I$, $|X_i \cap Y| \leqslant |X_i| \leqslant |A|$. Therefore, $Y \in F(A, B)$.

If $Z \sim X$, and $f : X \to Z$ is a bijective function, then f induces a partition $P_Z = \{f[X_i] : i \in I\}$ of Z, that respects the properties of P_X making Z an element of $\mathscr{C}$.

A similar argument is given for the other instances of $\mathscr{C}$. ∎

By Proposition 2.2.3, if we want to prove that $F(A, B)$, $F(< A, B)$, $F(A, < B)$, or $F(< A, < B)$ is a finiteness class, we only need to prove that every finite set is in it, and that ω does not.

Proposition 2.2.4. *Let ϕ, ϕ', ψ and ψ' be formulas with fixed parameters. If $\phi(x) \to \phi(x)$ for every x and $\psi(x, y) \to \psi'(x, y)$ for every x and y, then $F(\phi, \psi) \subseteq F(\phi', \psi')$.*

Proof. This follows directly from the definition of $F(\phi, \psi)$. ∎

Corollary 2.2.5. *If A, A', B and B' are sets such that $A \subseteq A'$ and $B \subseteq B'$, then $F(A, B) \subseteq F(A', B')$.*

Proposition 2.2.6. *The class $F(A, B)$ is a finiteness class if and only if both A and B are non-empty sets, either A is infinite or B is, and B is D-finite.*

Proof. We are going to begin assuming that $F(A, B)$ is a finiteness class. Consequently, every finite set belongs to it, including some non-empty set. Thus, A and B cannot be empty: for the former, any partition of a not empty set is not empty, and for the later, if B is empty then the partition of every set in $F(A, B)$ is just $\{\emptyset\}$ and this can not be a partition for a not empty set.

If A and B are finite sets, then a finite set with at least $|A||B| + 1$ elements can not belongs to $F(A, B)$. This contradicts the assumption that $F(A, B)$ contains every finite set. Finally, if B is D-infinite, then $\omega \in F(A, B)$, as the partition $\{\omega\}$ satisfies the required properties. However, this last assertion contradicts that $F(A, B)$ is a finiteness class. Therefore, B must be D-finite.

Now, we are going to assume that A and B are non-empty sets, one of them is infinite and B is D-finite. We will show first that every finite set is in $F(A, B)$. Let $Y = \{y_1, \ldots, y_n\}$, we have two cases from the hypothesis: if A is infinite, then we just take the partition $\{\{y_1\}, \ldots, \{y_n\}\}$; and if B is infinite, we just take the partition $\{Y\}$. In any case we get $Y \in F(A, B)$. Finally, to prove that $\omega \notin F(A, B)$, note that any infinite partition P_ω of ω has a choice function induced by the well order, and any finite partition has an infinite countable element, which does not respect the condition imposed by B. ∎

Corollary 2.2.7. *The class $F(\kappa, n)$ is a finite class for every cardinal κ and $n \in \omega \setminus \{0\}$.*

Proposition 2.2.8. *The class $F(A, < B)$ is a finite class if and only if A is not an empty set, $|B| > 1$; either A is infinite, or B is; and B is D-finite or $|B| = \omega$.*

Proof. The proof is essentially an adaptation of the proof of proposition 2.2.6.

∎

Corollary 2.2.9. *The class $F(\kappa, < \omega)$ is a finiteness class for every cardinal κ.*

2.3 Classes $F(\kappa, < \omega)$

Proposition 2.3.1. *For every cardinals κ and λ, such that $\kappa \leqslant \lambda$, we have that $F(\kappa, < \omega) \subseteq F(\lambda, < \omega)$.*

Proof. By proposition 2.2.4 ∎

Given any cardinal κ, it is consistent that for every $\lambda < \kappa$, $F(\lambda, < \omega) \subsetneq F(\kappa, < \omega)$. Additionally, it is consistent that for every cardinal κ, there exists X S-finite such that $X \notin F(\kappa, < \omega)$. See corollaries 3.2.2 and 3.3.5.

The following results relate the finiteness classes $F(\kappa, < \omega)$ with the choice statements $C(\kappa, < \omega)$.

Proposition 2.3.2. *1. For every cardinal κ, $C(\kappa, < \omega)$ implies $F(\kappa, < \omega) = FIN$.*

2. The König's Lemma is equivalent to $F(\omega, < \omega) = FIN$.

Proof. We begin proving (1). Let κ be a cardinal and assume $C(\kappa, < \omega)$. Let X be a set in $F(\kappa, < \omega)$. If X is not infinite, then it has an infinite partition $\{X_\alpha : \alpha < \kappa\}$ without infinite partial choice function, but it is a contradiction because we have $C(\kappa, < \omega)$.

Now, we continue with proving (2). We will prove more specifically that $C(\omega, < \omega)$ is equivalent to $F(\omega, < \omega) = \text{FIN}$. We have that $C(\omega, < \omega)$ implies $F(\omega, < \omega) = \text{FIN}$ from the last point. Thus, we only need to prove the backward implication.

We assume $F(\omega, < \omega) = \text{FIN}$. If $\{A_n : n \in \omega\}$ is a disjoint family with non-empty finite elements, then $X = \bigcup_{n \in \omega} A_n$ is infinite. Thus, $\{A_n : n \in \omega\}$ needs to have an infinite partial choice function. Therefore, every countable disjoint family has an infinite partial choice function; equivalently, they have a choice function (see the introduction of [2]).

∎

The second point of the last proposition give us the backward implication of the first point in the case $\kappa = \omega$. But, we do not have this for $\kappa > \omega$ (see corollary 3.2.4).

Negations of the Axiom of Choice can lead to some paradoxical results. With respect to our finiteness classes of partitions, we can ask whether it is consistent for the existence of sets in both classes $F(\kappa, < \omega)$ and $F(\lambda, < \omega)$, with $\kappa < \lambda$, having partitions of both cardinalities. The answer is no.

Proposition 2.3.3. *For every pair of cardinals κ and λ, with $\kappa < \lambda$, there is no set X having a partition of size κ consisting of finite sets and another partition of size λ consisting of finite sets.*

Proof. Assume that there exists a set X with partitions $\{P_\alpha : \alpha < \kappa\}$ and $\{Q_\alpha : \alpha < \lambda\}$, consisting of finite sets.

For every $\alpha < \lambda$, we define $r(\alpha) = \{\beta \in \kappa : Q_\alpha \cap P_\beta \neq \emptyset\}$. Since every Q_α is finite, it follows that $r(\alpha)$ is finite.

Consider the equivalency relation in λ:

$$\forall \gamma, \delta < \lambda : \gamma \sim \delta \text{ if and only if } r(\gamma) = r(\delta).$$

The equivalence classes of the former relation are finite subsets of λ characterized by finite subsets of κ. Then $|\lambda/\sim| \leqslant \kappa$, because $|[\kappa]^{<\omega}| = \kappa$ (see [3],theorem 4.19). This is a contradiction because $|\lambda/\sim| = \lambda$ (by proposition 1.3.7).

■

The last proposition avoids the paradoxical possibility of having partitions with different size consisting of finite sets, for the same set. However, this applies only to well-orderable partitions. Without this condition, such a paradoxical situation is consistent. See corollary 3.3.8.

Chapter 3

Consistency Proofs

3.1 Permutation Models

For a detailed study of the fundamentals of permutation models, refer to Chapter 3 in [6] and Chapter 7 in [3]. We will give a practical resume.

A permutation model is a model for Set Theory with Atoms (ZFA). In this theory, we retain the standard axioms of set theory but introduce modifications that allow for the existence of atoms. Atoms are objects within the theory that do not have elements (cannot appear on the right side of the symbol $\in$). The set of atoms will be denoted as $\mathcal{A}$.

Let X be a set. We define recursively on the class of ordinals

$$\begin{aligned} P^0(X) &= X, \\ P^{\alpha+1}(X) &= P(P^\alpha(X)) \text{ for every ordinal } \alpha, \\ P^\gamma(X) &= \bigcup_{\beta<\gamma} P^\beta(X) \text{ for every limit ordinal } \gamma. \end{aligned}$$

and $WF(X) = \bigcup_{\alpha \in Ord} P^\alpha(X)$.

In standard Set Theory (ZF and ZFC), by the Axiom of Foundation, we have that $WF(\emptyset)$ is the class of all sets. But, in Set Theory with Atoms (ZFA), this role is played by $WF(\mathcal{A})$. In other words, ZFA is "founded on the set of atoms".

In this chapter, we are going to operate within set theory with atoms and the axiom of choice (ZFA+AC). Additional consistent statements are often introduced, such as fixing the cardinality of $\mathcal{A}$.

We call $WF(\emptyset)$ as the kernel.

We say that $\sigma : \mathcal{A} \to \mathcal{A}$ is a permutation of $\mathcal{A}$ if it is bijective. For every σ permutation of $\mathcal{A}$, we extend it as a bijective function $\sigma : V \to V$ on the universe (we denote σ and its extension with the same symbol) recursively: for every set X

$$\sigma(X) = \{\sigma(x) : \ x \in X\}.$$

We fix G as subgroup of $S_{\mathcal{A}}$. We say (and fix it) that $\mathcal{I}$ is a *normal ideal* if it is a family of subsets of $\mathcal{A}$ that satisfies the following:

1. $\emptyset \in \mathcal{I}$;
2. for every $A \in \mathcal{I}$ and $B \subseteq A$, we have $B \in \mathcal{I}$;
3. for every $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$;
4. for every $\sigma \in G$ and $A \in \mathcal{I}$, we have $\sigma(A) \in \mathcal{I}$;
5. for every $a \in \mathcal{A}$, we have $\{a\} \in \mathcal{I}$.

For every $E \subseteq \mathcal{A}$ we denote

$$\mathrm{fix}_G(E) = \{\sigma \in G : (\forall a \in E)\sigma(a) = a\}.$$

For every set X we denote

$$\mathrm{sym}_G(X) = \{\sigma \in G : \sigma(X) = X\}.$$

We will write just $\mathrm{fix}(E)$ and $\mathrm{sym}(X)$ because the subgroup will be present implicitly. Both of them are subgroups of G.

The permutation model defined (recursively) by G and $\mathcal{I}$ is

$$\mathbb{M} = \{X : \exists E \subseteq \mathcal{A} \text{ in } \mathcal{I} \text{ such that } \mathrm{fix}(E) \subseteq \mathrm{sym}(X) \text{ and } X \subseteq \mathbb{M}\}.$$

$\mathbb{M}$ is a transitive model of ZFA, $\mathcal{A} \in \mathbb{M}$ and $WF(\emptyset) \subseteq \mathbb{M}$ (see Theorem 4.1 in [6]).

We will use only two instances of $\mathcal{I}$: $\mathcal{I} = [\mathcal{A}]^{<\omega}$ (the ideal of finite subsets of $\mathcal{A}$) and $\mathcal{I} = \{A \subseteq \mathcal{A} : A \text{ is countable}\}$. Then we will say that $\mathbb{M}$ is the permutation model with finite (countable, in the second case) support defined by G. In this case, if $x \in \mathbb{M}$ and E is a finite (countable) subset of $\mathcal{A}$, such that $\mathrm{fix}(E) \subseteq \mathrm{sym}(x)$, we say that E is a support of x. Every element of $\mathbb{M}$ has a support.

Although the consistency proofs through permutation models are relative to Set Theory with Atoms, there are results that transfer these consistency proofs to ZF. The one that we will use in this work is the Jech-Sochor Embedding Theorem, from [5], but the reader can see [6] (chapter 6) and [3] (theorem 17.2). This theorem state that some statements from permutation models can be simulated by symmetric models. A symmetric model is a model of ZF. We state a practical corollary.

Lemma 3.1.1 (Corollary 17.3 [3])**.** *Let α be an ordinal and let ϕ be a sentence of the form $\exists X \psi(X, \alpha)$, where the only quantifiers we allow in ψ are the restricted quantifiers $\exists u \in P^\alpha(X)$ and $\forall u \in P^\alpha(X)$. If $\mathbb{M} \models$ ZFA is a permutation model in which AC holds in the kernel and $\mathbb{M} \models \phi$, then there exists a symmetric model $\hat{\mathbb{M}} \models$ ZF such that $\hat{\mathbb{M}} \models \phi$.*

Although we could use symmetric models for our consistency proofs, it could require knowledge about Forcing theory, and permutation models can be used more directly.

The corresponding statements for the consistency results in this chapter satisfy the hypotheses of Lemma 3.1.1.

3.2 The κ-second Fraenkel Model

We will consider the set of atoms with a partition of cardinality κ, consisting of sets with two elements:

$$\mathcal{A} = \bigcup_{\alpha<\kappa} \{a_\alpha, b_\alpha\}.$$

Denote $P = \{\{a_\alpha, b_\alpha\} : \alpha < \kappa\}$.

Let $G = \{\sigma \in S_{\mathcal{A}} : (\forall \alpha < \kappa), \sigma\{a_\alpha, b_\alpha\} = \{a_\alpha, b_\alpha\}\}$[1] and $\mathbb{M}$ be the permutation model with finite supports defined by G. We refer to this model as the κ-second Fraenkel Model.

Note that $P \in \mathbb{M}$. This is because $P \subseteq \mathbb{M}$, since every element of P is a pair of atoms; and, by definition, $\text{fix}(\emptyset) = G = sym(P)$.

Theorem 3.2.1. *Let κ and λ be infinite cardinals, with $\kappa > \lambda$, and $\mathbb{M}$ be the κ-second Fraenkel Model. Then $\mathbb{M}$ models that $\mathcal{A} \in F(\kappa, <\omega)$ and $\mathcal{A} \notin F(\lambda, <\omega)$.*

Proof. Fix κ and λ as infinite cardinals, with $\kappa > \lambda$, and let $\mathbb{M}$ be the κ-second Fraenkel model.

We will first verify that the corresponding partition P of $\mathcal{A}$ witnesses $\mathcal{A} \in F(\kappa, <\omega)$.

Suppose that P has an infinite partial choice function. Since P is indexed by κ, there exists $f; \kappa \to \mathcal{A}$ in $\mathbb{M}$, such that, for every $\alpha < \kappa$, $f(\alpha) = a_\alpha$ or $f(\alpha) = b_\alpha$. Then there exists a finite subset E of $\mathcal{A}$ such that $\text{fix}(E) \subseteq \text{sym}(f)$. As E is finite, there exists $\beta \in \kappa$ such that $a_\beta, b_\beta \notin E$. Consider the permutation of $\mathcal{A}$

$$\pi(x) = \begin{cases} b_\beta & \text{if } x = a_\beta, \\ a_\beta & \text{if } x = b_\beta, \\ x & \text{other cases.} \end{cases}$$

This permutation belongs to $\text{fix}(E)$. Without loss of generality, let $f(\beta) = a_\beta$. Then $(\beta, a_\beta) \in f$, but $\pi(\beta, a_\beta) = (\beta, b_\beta)$, following that $(\beta, b_\beta) \in f$, a contradiction. Therefore, such a function f cannot belong to $\mathbb{M}$. This implies that $\mathbb{M}$ models $\mathcal{A} \in F(\kappa, <\omega)$.

By Proposition 2.3.3, $\mathbb{M}$ does not model that $\mathcal{A} \in F(\lambda, <\omega)$. ∎

Corollary 3.2.2. *For every cardinal κ, it is consistent that for every cardinal $\lambda < \kappa$, $F(\lambda, <\omega) \subsetneq F(\kappa, <\omega)$.*

Now we will consider $\mathbb{M}$ exactly as before but with numerable support. We refer to this model as the κ-second Fraenkel Model with numerable support.

Theorem 3.2.3. *Let $\kappa > \omega$ a cardinal number. The κ-second Fraenkel Model with numerable support $\mathbb{M}$ models that $F(\kappa, <\omega) = FIN$ and P is a counter example for $C(\kappa, <\omega)$.*

Proof. Fix κ and $\mathbb{M}$ as in the hypotheses.

First we will prove that $\mathbb{M}$ models $F(\kappa, <\omega) = \text{FIN}$.

[1] We will usually write σx instead of $\sigma(x)$, for $\sigma \in S_{\mathcal{A}}$ and x a set or an atom.

Let X be an infinite set in $\mathbb{M}$, and $Q = \{X_\alpha : \alpha < \kappa\}$ a partition of X, such that $Q \in \mathbb{M}$. For this partition, there exists a countable partial function $g; \kappa \to X$ in V, such that, for every $\alpha \in \mathrm{dom(g)}$, $g(\alpha) \in X_\alpha$ (it exists because we assume the Axiom of Choice out of the model). We numerate the domain of g as $\{\alpha_n : n \in \omega\}$.

As $X \subseteq \mathbb{M}$, then for every $n \in \omega$ there exists some E_n, a countable subset of $\mathcal{A}$, such that $\mathrm{fix}(E_n) \subseteq \mathrm{sym}(g(\alpha_n))$. Then we have that $E = \bigcup_{n\in\omega} E_{\alpha_n}$ is countable. As $\mathrm{fix}(E) \subseteq \mathrm{sym}(g)$, then we have that $g \in \mathbb{M}$. Therefore $\mathbb{M}$ models $F(\kappa, <\omega) = \mathrm{FIN}$.

Now we will verify that P witnesses that $\mathbb{M}$ does not model $C(\kappa, <\omega)$.

Suppose that there exists a choice function $f : \kappa \to \mathcal{A}$ for the partition P. Let $E \subseteq \mathcal{A}$ be a countable support of f. Since $\kappa > \omega$ there exists $\alpha < \kappa$ such that $a_\alpha, b_\alpha \notin E$. The remainder of the proof is analogous to the first part of the proof of theorem 3.2.1. ■

Corollary 3.2.4. *For every cardinal $\kappa > \omega$, it is consistent that $F(\kappa, <\omega) = FIN$ with the negation of $C(\kappa, <\omega)$.*

3.3 The non-orderable second Fraenkel Model

We will consider the set of atoms with an infinite partition into sets of two elements:

$$\mathcal{A} = \bigcup_{i\in J} \{a_i, b_i\}.$$

Where J is any infinite set. Denote $P = \{\{a_i, b_i\} : i \in J\}$.

Let $G = \{\sigma \in S_A : (\forall i \in J)(\exists j \in J)\sigma\{a_i, b_i\} = \{a_j, b_j\}\}$ and $\mathbb{M}$ be the permutation model with finite supports defined by G. We refer to this model as the non-orderable second Fraenkel Model.

Analogously to the κ-second Fraenkel Model, here we have that $P \in \mathbb{M}$.

The name of the model is due to the following observation.

Proposition 3.3.1. *The non-orderable second Fraenkel Model models that the corresponding partition P is infinite and non-well-orderable.*

Proof. Assume the existence of some well-order relation R on P. Let $E \subseteq \mathcal{A}$ be a finite support for R. Since E is finite, there is some i and j in J such that $a_i, b_i, a_j, b_j \notin E$. Consider the permutation of $\mathcal{A}$

$$\pi(x) = \begin{cases} a_i & \text{if } x = a_j, \\ b_i & \text{if } x = b_j, \\ a_j & \text{if } x = a_i, \\ b_j & \text{if } x = b_i, \\ x & \text{other cases.} \end{cases}$$

This permutation belongs to $\mathrm{fix}(E)$. Without loss of generality, let $(\{a_i, b_i\}, \{a_j, b_j\}) \in R$. But we have that $\pi(\{a_i, b_i\}, \{a_j, b_j\}) = (\{a_j, b_j\}, \{a_i, b_i\})$, then $(\{a_j, b_j\}, \{a_i, b_i\})$ belongs to R, but this is a contradiction. Therefore $\mathbb{M}$ models P is not well-orderable. ■

Theorem 3.3.2. *The non-orderable second Fraenkel Model, models that the set of atoms does not have a well orderable infinite partition.*

Proof. We prove it by contradiction. Let κ be an infinite cardinal. Consider a partition $Q = \{A_\alpha : \alpha < \kappa\}$ of $\mathcal{A}$, where A_α is non-empty for every $\alpha < \kappa$. If Q and its indexing $e : \kappa \to Q$ (i.e, $e(\alpha) = A_\alpha$ for each $\alpha < \kappa$) belong to $\mathbb{M}$, then there exists some $E \subseteq \mathcal{A}$ finite support of Q and e. Without loss of generality, we have that, for every $i \in J$, if $a_i \in E$ or $b_i \in E$, then $\{a_i, b_i\} \subseteq E$.

As E is finite, there exists $\beta < \kappa$, such that $A_\alpha \cap E = \emptyset$ for every $\alpha \geqslant \beta$.

Let $j \in J$ such that $A_\beta \cap \{a_j, b_j\} \neq \emptyset$. Then, by the selection of E and β, we have $E \cap \{a_j, bj\} = \emptyset$. Without loss of generality we consider $a_j \in A_\beta$.

Consider any $\alpha > \beta$, and $x \in A_\alpha$. Without loss of generality we have $x = a_l$ for some $l \in J$. By the selection of E, $\{a_l, b_l\} \cap E = \emptyset$. Observe that $a_l \notin A_\beta$.

Consider the permutation π that interchanges a_j with a_l and b_j with b_l, and fixes the other atoms. We have $\pi \in G$. Furthermore, $\pi \in \operatorname{fix}(E)$ because $\{a_j, b_j\} \cap E = \emptyset$ and $\{a_l, b_l\} \cap E = \emptyset$. Then, $\pi \in \operatorname{sym}(e)$, therefore $\pi(\beta, A_\beta) \in e$, but $\pi(\beta, A_\beta) = (\beta, \pi A_\beta) \neq (\beta, A_\beta)$, because $\pi A_\beta \neq A_\beta$, since $a_l = \pi(a_j)$ does not belong to A_β. This last conclusion contradicts that e is a function. ∎

Corollary 3.3.3. *It is consistent that there exists a set that has an infinite partition of finite sets, but it does not have a well orderable partition.*

Theorem 3.3.4. *The non-orderable second Fraenkel Model $\mathbb{M}$ models that $\mathcal{A}$ is S-finite, yet for every infinite cardinal κ, $\mathcal{A}$ does not belong to $F(\kappa, <\omega)$.*

Proof. The proof that $\mathbb{M}$ models $\mathcal{A}$ as S-finite is analogous to the proof that the κ-second Fraenkel Model models $\mathcal{A} \in F(\kappa, <\omega)$ (Theorem 3.2.1), with $P = \{\{a_i, b_i\} : i \in J\}$ as its respective partition.

Let κ be an infinite cardinal. We conclude that $\mathbb{M}$ models $\mathcal{A} \notin F(\kappa, <\omega)$ because, by lemma 3.3.2, we cannot have any partition of size κ for $\mathcal{A}$ in $\mathbb{M}$. ∎

Corollary 3.3.5. *It is consistent that there exists an S-finite set X such that for every cardinal κ, $X \notin F(\kappa, <\omega)$.*

We employ the non-orderable second Fraenkel Model for another consistency proof, fixing $J = \omega$.

Proposition 3.3.6. *The set $R = \{\{a_n, b_n, \{a_n, b_n\}\} : n \in \omega\}$ belongs to the non-orderable second Fraenkel Model $\mathbb{M}$ with $J = \omega$.*

Proof. We prove that $\emptyset$ is a support for R. If π is any permutation of $\mathcal{A}$ in G, then for every $n \in \omega$ exists $m \in \omega$ such that $\pi(\{a_n, b_n, \{a_n, b_n\}\}) = \{a_m, b_m, \{a_m, b_m\}\}$, because $\pi(\{a_n, b_n\}) = \{a_m, b_m\}$. Therefore $R \in \mathbb{M}$. ∎

Theorem 3.3.7. *Consider $X = \mathcal{A} \cup P$, and consider $R = \{\{a_n, b_n, \{a_n, b_n\}\} : n \in \omega\}$ and $L = \{\{x\} : x \in X\}$ the partitions of X. The non-orderable second Fraenkel Model $\mathbb{M}$ models that $|R| < |L|$.*

Proof. We verify that $\mathbb{M}$ models $|R| \leqslant |L|$ because the empty set serves as a support for the injective function $f : R \to L$, where f maps $\{a_n, b_n, \{a_n, b_n\}\}$ to $\{\{a_n, b_n\}\}$ for every $n \in \omega$. This follows from the fact that for every $\pi \in G$ and $n \in \omega$, there exists $m \in \omega$ such that $\pi(\{a_n, b_n, \{a_n, b_n\}\}, \{\{a_n, b_n\}\}) = (\{a_m, b_m, \{a_m, b_m\}\}, \{\{a_m, b_m\}\})$.

Now we will prove that $\mathbb{M}$ models that $|L| \nleqslant |R|$, by contradiction.

We assume that there exists an injective function $g : L \to R$ in $\mathbb{M}$. Let $E \subseteq \mathcal{A}$ be a finite support for g and $n, m \in \omega$ such that $\{a_n, b_n\} \cap E = \emptyset$ and $g(\{a_n\}) = \{a_m, b_m, \{a_m, b_m\}\}$.

We consider the permutation of $\mathcal{A}$

$$\pi(x) = \begin{cases} a_n & \text{if } x = b_n, \\ b_n & \text{if } x = a_n, \\ x & \text{other cases.} \end{cases}$$

This permutation belongs to $\mathrm{fix}(E)$. In both cases, whether $n = m$ or $n \neq m$, $\pi\{a_m, b_m, \{a_m, b_m\}\} = \{a_m, b_m, \{a_m, b_m\}\}$. Then

$$\pi(\{a_n\}, \{a_m, b_m, \{a_m, b_m\}\}) = (\{b_n\}, \{a_m, b_m, \{a_m, b_m\}\}),$$

therefore $(\{b_n\}, \{a_m, b_m, \{a_m, b_m\}\})$ belongs to g. However, this leads to a contradiction because g is injective.

■

Corollary 3.3.8. *It is consistent for there to exist sets A, B, and C such that $|B| < |C|$ and A has partitions of both cardinalities consisting of finite sets.*

Bibliography

[1] J. W. Degen. Some aspects and examples of infinity notions. *Math. Logic Quart.*, 40(1):111–124, 1994. URL: https://doi-org.pbidi.unam.mx:2443/10.1002/malq.19940400116, doi:10.1002/malq.19940400116. 3

[2] T. E. Forster and J. K. Truss. Ramsey's theorem and König's lemma. *Arch. Math. Logic*, 46(1):37–42, 2007. URL: https://doi-org.pbidi.unam.mx:2443/10.1007/s00153-006-0025-z, doi:10.1007/s00153-006-0025-z. 3, 9

[3] Lorenz J. Halbeisen. *Combinatorial set theory*. Springer Monographs in Mathematics. Springer, London, 2012. With a gentle introduction to forcing. URL: https://doi-org.pbidi.unam.mx:2443/10.1007/978-1-4471-2173-2, doi:10.1007/978-1-4471-2173-2. 1, 2, 4, 5, 10, 11, 12

[4] Horst Herrlich. The finite and the infinite. *Appl. Categ. Structures*, 19(2):455–468, 2011. URL: https://doi-org.pbidi.unam.mx:2443/10.1007/s10485-009-9212-5, doi:10.1007/s10485-009-9212-5. 3

[5] T. Jech and A. Sochor. Applications of the Θ-model. *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.*, 14:351–355, 1966. 12

[6] Thomas J. Jech. *The axiom of choice*. Studies in Logic and the Foundations of Mathematics, Vol. 75. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. 11, 12

[7] K. Kunen. *Set Theory*. College Publications, 2011. 5

[8] K. Kunen. *Set theory an introduction to independence proofs*. Elsevier, 2014. 5

[9] N. Weaver. *Forcing For Mathematicians*. World Scientific Publishing Company, 2014. URL: https://books.google.com.mx/books?id=Fke7CgAAQBAJ. 1