Strong Failures of Higher Analogs of Hindman's Theorem

David Fernández-Bretón
(joint work with Assaf Rinot)

djfernan@umich.edu
http://www-personal.umich.edu/~djfernan

Department of Mathematics,
University of Michigan

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$$\mathrm{FS}(X) = \{x_1 + \cdots + x_n \big| n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X \text{ are distinct}\}.$$

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Theorem (Galvin/Glazer/Hindman)

For every commutative cancellative semigroup G and every colouring $c : G \longrightarrow 2$ with two colours, there exists an infinite $X \subseteq G$ such that $\mathrm{FS}(X)$ is monochromatic.

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In all known proofs of this result, the set X is constructed by means of a recursion with ω steps.

Question

Is it possible to find, given a colouring $c : G \longrightarrow 2$ *of an uncountable commutative cancellative semigroup, an uncountable* X *with* $\mathrm{FS}(X)$ *monochromatic?*

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Theorem

For every uncountable commutative cancellative semigroup G there exists a colouring $c : G \longrightarrow 2$ such that whenever $X \subseteq G$ is uncountable, $\mathrm{FS}(X)$ is not monochromatic.

Our key algebraic tool to treat these problems is the following result:

Theorem

Let G be any commutative cancellative semigroup of cardinality $\kappa > \omega$. Then there are countable abelian groups G_α, $\alpha < \kappa$, such that G embeds into

$$\bigoplus_{\alpha<\kappa} G_\alpha = \left\{ x \in \prod_{\alpha<\kappa} G_\alpha \middle| x(\alpha) = 0 \text{ for all but finitely many } \alpha < \kappa \right\}.$$

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Note that, if $c : \bigoplus_{\alpha<\kappa} G_\alpha \longrightarrow 2$ is a "bad" colouring, then so is $c \upharpoonright G$. Thus from now on we will assume without loss of generality that $G = \bigoplus_{\alpha<\kappa} G_\alpha$, where each G_α is countable.

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Given $x \in \bigoplus_{\alpha<\kappa} G_\alpha$, we will define the **support** of x to be

$$\mathrm{supp}(x) = \{\alpha < \kappa | x(\alpha) \neq 0\} \in [\kappa]^{<\omega}.$$

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We denote the statement above by $G \nrightarrow [\omega_1]^{\mathrm{FS}}_{\omega}$ (recall the square-bracket notation for higher analogs of Ramsey's theorem).

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Under which circumstances can we increase the number of colours?

Lemma

If G is commutative cancellative of cardinality $\kappa > \omega$, then there exists a $d : G \longrightarrow [\kappa]^{<\omega}$ such that for every uncountable $X \subseteq G$, there exists an $A \subseteq \kappa$ with $|A| = |X|$ and $[A]^{<\omega} \subseteq d[\mathrm{FS}(X)]$.

Theorem

For all cardinals λ, κ, θ with $\omega_1 \leq \lambda \leq \kappa$, the following are equivalent:

- $\kappa \nrightarrow [\lambda]^{<\omega}_{\theta}$,
- $G \nrightarrow [\lambda]^{\mathrm{FS}}_{\theta}$ *for every commutative cancellative semigroup of cardinality* κ,
- $G \nrightarrow [\lambda]^{\mathrm{FS}}_{\theta}$ *for some commutative cancellative semigroup of cardinality* κ.

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Corollary

It is consistent with ZFC *(e.g. if $\mathbf{V} = \mathbf{L}_{\kappa}$ where κ is the least inaccessible) that for every commutative cancellative G, $G \nrightarrow [\theta]^{\mathrm{FS}}_{\theta}$ holds for every uncountable θ.*

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Corollary

Modulo large cardinals it is consistent with ZFC *(e.g. after forcing to add κ Cohen reals, where κ is an ω_1-Erdős cardinal in the ground model) that $\mathbb{R} \rightarrow [\omega_1]^{\mathrm{FS}}_{\omega_1}$.*

We now attempt to obtain results along the same lines, but replacing FS with FS_2, where for $X \subseteq G$ we define $\mathrm{FS}_2(X) = \{x + y \,|\, x, y \in X \text{ are distinct}\}$.

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Definition

Given cardinals $\kappa \geq \theta$, the symbol $S^*(\kappa, \theta)$ will denote the following statement: there exists a colouring $d : [\kappa]^{<\omega} \longrightarrow \theta$ such that, if $\mathcal{X}, \mathcal{Y} \subseteq [\kappa]^{<\omega}$ are families satisfying $|\mathcal{X}| = |\mathcal{Y}| = \kappa$, then for every $\delta < \theta$ there are elements $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $d(z) = \delta$ whenever $x \bigtriangleup y \subseteq z \subseteq x \cup y$.

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Theorem

Let $\kappa \geq \theta \geq \omega_1$. If $S^*(\kappa, \theta)$ holds, then for every commutative cancellative G with $|G| = \kappa$, $G \nrightarrow [\kappa]^{\mathrm{FS}_2}_\theta$.

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Theorem

Let $\kappa \geq \theta \geq \omega_1$. If $S^*(\kappa, \theta)$ holds, then for every commutative cancellative G with $|G| = \kappa$, $G \nrightarrow [\kappa]_\theta^{\mathrm{FS}_2}$. In fact, the following stronger statement holds: there exists a colouring $c : G \longrightarrow \theta$ such that for every two sets $X, Y \subseteq G$ with $|X| = |Y| = \kappa$, the sumset $X + Y = \{x + y \big| x \in X \text{ and } y \in Y\}$ attains all possible colours (that is, $c[X + Y] = \theta$).

Recall that the combinatorial principle $\mathrm{Pr}_1(\kappa, \lambda, \theta, \chi)$ states the existence of a colouring $c : [\kappa]^2 \longrightarrow \theta$ such that, whenever $\mathcal{X} \subseteq [\kappa]^{<\chi}$ has size λ and is pairwise disjoint, for all $\delta < \theta$ we can find two distinct $x, y \in \mathcal{X}$ such that $c[x \times y] = \{\delta\}$.

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Fact

If $\mathrm{cf}(\kappa) = \kappa > \omega_1$ *admits a nonreflecting stationary set, then* $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \omega)$ *holds (for example, if* $\kappa = \lambda^+$ *for* $\lambda = \mathrm{cf}(\lambda) \geq \omega_1$*).*

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If $\kappa = \mathrm{cf}(\kappa) \geq \omega_1$ and $\theta \leq \kappa$, then $\mathrm{Pr}_1(\kappa, \kappa, \theta, \omega)$ implies $S^*(\kappa, \theta)$. In particular, if $\mathrm{Pr}_1(\kappa, \kappa, \theta, \omega)$ holds then $G \nrightarrow [\kappa]^{\mathrm{FS}_2}_{\theta}$ whenever $|G| = \kappa$.

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Theorem

If κ is singular, then $S^*(\mathrm{cf}(\kappa), \theta)$ implies that $S^*(\kappa, \theta)$. In particular, if κ is such that $\mathrm{cf}(\kappa) > \omega_1$ admits a nonreflecting stationary subset, then $S^*(\kappa, \mathrm{cf}(\kappa))$ holds, and consequently, $G \nrightarrow [\kappa]^{\mathrm{FS}_2}_{\mathrm{cf}(\kappa)}$ for every commutative cancellative G of cardinality κ.

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Theorem

Let κ be an infinite cardinal satisfying $2^{<\kappa} = \kappa$. Then for every commutative cancellative G such that $\mathrm{cf}(|G|) = \mathrm{cf}(2^\kappa)$, we have that $G \not\rightarrow [|G|]^{\mathrm{FS}_2}_\omega$. In particular, the statement $\mathbb{R} \not\rightarrow [\mathfrak{c}]^{\mathrm{FS}_2}_\omega$ holds.

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Theorem

If $\mathfrak{c}$ is regular, and not weakly compact in $\mathbf{L}$, then $\mathbb{R} \nrightarrow [\mathfrak{c}]^{\mathrm{FS}_2}_{\mathfrak{c}}$. On the other hand, after adding κ Cohen reals to a model where κ is weakly compact, we obtain $\mathbb{R} \rightarrow [\mathfrak{c}]^{\mathrm{FS}_2}_{\omega_1}$.