Ultrafilters on Semigroups and some of their Properties

(Strong Summability, Sparseness, Idempotence, etc.)

David J. Fernández Bretón

Department of Mathematics and Statistics
York University

Dissertation Subject Oral Examination
December 18th, 2012

The Stone-Čech compactification of a discrete abelian semigroup S is the set βS of ultrafilters on S, where every $x \in S$ is identified with

$$\{A \subseteq S \big| x \in A\},$$

and basic open sets are those of the form

$$\bar{A} = \{p \in \beta S \big| A \in p\}.$$

Then these sets are actually clopen, and $\bar{A}$ is really the closure in βS of the set A.
The group operation $+$ on S is also extended by the formula

$$p + q = \{A \subseteq S \big| \{x \in S \big| \{y \in S \big| x + y \in A\} \in q\} \in p\}$$

which turns βS into a right semitopological semigroup, meaning that for each $p \in \beta S$ the mapping $(\cdot) + p : \beta S \longrightarrow \beta S$ is continuous (note that the extended operation $+$ need not be commutative).

If S is a semigroup, and $\vec{x} = \langle x_n \big| n < \omega \rangle$ is a sequence of elements of S, then we denote the *set of finite sums of the sequence* $\vec{x}$ by:

$$\mathrm{FS}(\vec{x}) = \left\{ \sum_{n \in a} x_n \middle| a \in [\omega]^{<\omega} \setminus \{\varnothing\} \right\}.$$

Definition

Let S be a semigroup, and $p \in \beta S$.

- We say that p is **weakly summable** if for every $A \in p$ there exists a sequence $\vec{x}$ such that $\mathrm{FS}(\vec{x}) \subseteq A$.
- We say that p is **strongly summable** if it is weakly summable, and additionally, the above sequence $\vec{x}$ can be chosen in such a way that $\mathrm{FS}(\vec{x}) \in p$.

On abelian groups, strongly summable implies idempotent, which in turn implies weakly summable. However, the existence of a strongly summable ultrafilter on $(\omega, +)$ implies that of a P-point and hence cannot be established in ZFC.

The importance of these concepts stems from the following

Theorem (Hindman)

Let $\omega = \bigcup_{i<n} A_i$ *be a partition of* ω *into finitely many pieces. Then there exists a sequence* $\vec{x}$ *of natural numbers and an element* A_i *of the partition such that* $\mathrm{FS}(\vec{x}) \subseteq A_i$.

Proof.

Use the so-known *Ellis-Nukamura Lemma* to get an idempotent ultrafilter $p \in \beta\omega$. Then p chooses one element A_i of the partition, and since p must be weakly summable, the result follows. □

This provides an elegant proof of Hindman's finite sums theorem. It was actually Neil Hindman who first constructed strongly summable ultrafilters on ω, under CH, since at the time he was not aware of the Ellis-Nukamura Lemma, but knew that an idempotent ultrafilter would give him the desired result.

Strongly summable ultrafilters have some properties that not all idempotents have.

Theorem (Hindman-Strauss)

Let $p \in \beta\omega$ be a strongly summable ultrafilter, and let $q, r \in \omega^*$ be such that $q + r = r + q = p$. Then, $q, r \in \mathbb{Z} + p$.

Theorem (Hindman-Protasov-Strauss)

If G can be embedded in the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $p \in \beta G$ is strongly summable, then whenever $q, r \in G^* = \beta G \setminus G$ are such that $q + r = r + q = p$, it must be the case that $q, r \in G + p$.

By strengthening a bit the definition of strongly summable, Hindman, Protasov and Strauss were able to get a slightly stronger theorem.

Definition

An ultrafilter $p \in \beta G$ is **sparse** if for every $A \in p$ there exist two sequences $\vec{x} = \langle x_n \big| n < \omega \rangle$, $\vec{y} = \langle y_n \big| n < \omega \rangle$, where $\vec{y}$ is a subsequence of $\vec{x}$ such that $\{x_n \big| n < \omega\} \setminus \{y_n \big| n < \omega\}$ is infinite, $\mathrm{FS}(\vec{x}) \subseteq A$, and $\mathrm{FS}(\vec{y}) \in p$.

MA implies that there are sparse ultrafilters. And obviously every sparse ultrafilter will be strongly summable. But sparse ultrafilters have a stronger property.

Theorem (Hindman-Protasov-Strauss)

If G can be embedded in $\mathbb{T}$ and $p \in G^*$ is sparse, then whenever $q, r \in G^*$ are such that $q + r = p$, it must be the case that $q, r \in G + p$.

Theorem (Hindman-Steprāns-Strauss)

The semigroup $(\omega, +)$ has the property that every strongly summable ultrafilter on it is sparse. So does every subsemigroup of $\mathbb{T}$.

Theorem (Hindman-Steprāns-Strauss)

Let S be a countable subsemigroup of $\bigoplus_{n<\omega} \mathbb{T}$ and let p be a nonprincipal strongly summable ultrafilter on S. If

$$\left\{ x \in S \middle| \pi_{\min(x)}(x) \neq \frac{1}{2} \right\} \in p,$$

then p must be sparse (here $\min(x)$ denotes the minimum i such that $\pi_i(x)$ is nonzero).

Question (Hindman-Steprāns-Strauss)

Is it consistent with ZFC *that there exists a nonsparse strongly summable ultrafilter on* $\bigoplus_{n<\omega} \mathbb{Z}_2$*?*

Theorem (F.B.)

Let p *be a strongly summable ultrafilter on* $\bigoplus_{n<\omega} \mathbb{Z}_2$*. Then,* p *is sparse.*

Theorem (F.B.)

If there is an abelian cancellative semigroup S *and a nonsparse strongly summable ultrafilter on* S*, then there is one on* $\bigoplus_{n<\omega} \mathbb{Z}_{2^n}$*.*

Question

Is every strongly summable ultrafilter on any abelian group (equivalently, on $\bigoplus_{n<\omega} \mathbb{T}$) sparse?

Question

Is there (under suitable assumptions, such as MA*) a strongly summable ultrafilter on $\bigoplus_{n<\omega} \mathbb{Z}_2$ that is not additively isomorphic to a union ultrafilter?*

Question

Does the existence of a strongly summable ultrafilter on any abelian group imply that of a P-point?

Question

What happens when G is not abelian?

Question

Does the existence of a nondiscrete extremally disconnected group topology on $([\omega]^{<\omega}, \triangle)$ implies that of a strongly summable ultrafilter? What about a P-point? Is there a model with P-points but no strongly summable ultrafilters (say, on the Boolean group)?

In general, for a fixed $q \in \beta\omega$, the mapping $q + (\cdot) : \beta\omega \longrightarrow \beta\omega$ is not continuous. However, every P-point is a point of continuity of such a map for every $q \in \beta\omega$.

Question (Protasov)

Are there $p, q \in \omega^*$ such that p is not a P-point, yet it is a point of continuity of $q + (\cdot)$?

Conjecture (Steprāns)

There is a model of ZFC *in which there are no P-points, yet there is one (are many?) $p \in \omega^*$ that is a point of continuity of $q + (\cdot)$ for some (many?) $q \in \omega^*$.*

- Blass, A. and Hindman, N., On Strongly Summable Ultrafilters and Union Ultrafilters, Trans. Amer. Math. Soc. **304** No. 1 (1987), 83-99.
- Fernández Bretón, D., Every Strongly Summable Ultrafilter on $\bigoplus_{n<\omega} \mathbb{Z}_2$ is Sparse, preprint, 2012.
- Hindman, N., The existence of certain ultrafilters on $\mathbb{N}$ and a conjecture of Graham and Rothschild, Proc. Amer. Math. Soc. **36** (1972), 341-346.
- Hindman, N., Protasov, I. and Strauss, D., Strongly Summable Ultrafilters on Abelian Groups, Matem. Studii **10** (1998), 121-132.
- Hindman, N., Steprāns, J. and Strauss, D., Semigroups in which all Strongly Summable Ultrafilters are Sparse, New York J. Math. **18** (2012), 835-848.
- Hindman, N. and Strauss, D., Algebra in the Stone-Čech Compactification, de Gruyter Expositions in Mathematics 27, Walter de Gruyter, Berlin-New York, 1998.
- Krautzberger, P., On strongly summable ultrafilters, New York J. Math. **16** (2010), 629-649.