Variations on a theme: Ramsey's and Hindman's theorem

David J. Fernández-Bretón

~~University of Michigan~~ On the move

joint works with elements of the set $\{\varnothing, \text{J. Brot}, \text{M. Cao}, \text{S. H. Lee}, \text{A. Rinot}\}$

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What is Ramsey theory?

Theorem

Proof.

□

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In every party with at least 6 attendees, there are three of them that either mutually know each other or are mutually unknown to each other.

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Whenever we colour the edges of a complete graph with at least 6 vertices using two colours, there will necessarily be a monochromatic triangle.

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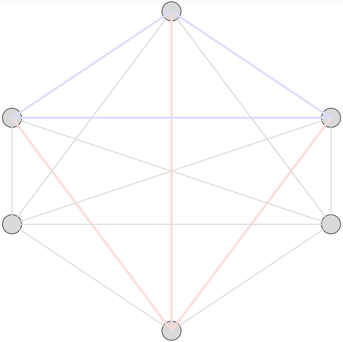

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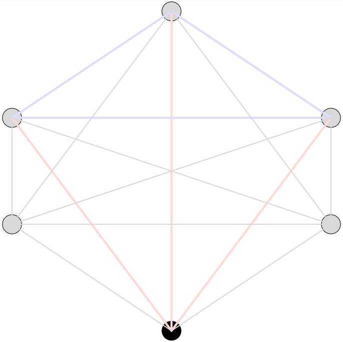

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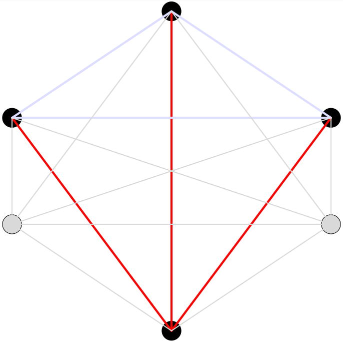

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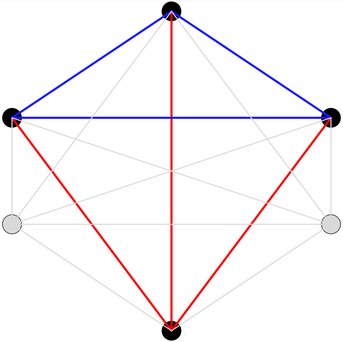

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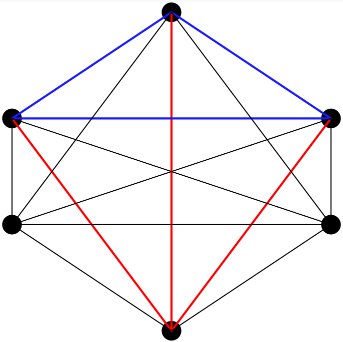

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Ramsey's theorem

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For every $n \in \mathbb{N}$ there exists an $R(n) \in \mathbb{N}$ such that for every coulouring $c : [R(n)]^2 \longrightarrow 2$ there exists an $X \subseteq R(n)$ with $|X| = n$ such that $|c``[X]^2| = 1$ (i.e. there is a monochromatic complete induced subgraph with n vertices).

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Theorem

For every colouring $c : [\omega]^2 \longrightarrow 2$ *there exists an infinite* $X \subseteq \omega$ *such that* $[X]^2$ *is monochromatic (i.e. there is an infinite monochromatic induced subgraph).*

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For every infinite abelian group G and every colouring $c : G \longrightarrow 2$, there exists an infinite $X \subseteq G$ such that $\mathrm{FS}(X)$ is monochromatic.

Theme 1: The Čech–Stone compactification, aka ultrafilters

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An **ultrafilter** over a set X is a family $u \in \mathfrak{P}(\mathfrak{P}(X))$ satisfying:

1. $(\forall A, B \subseteq X)(A \cap B \in u \iff (A \in u \wedge B \in u))$,
2. $(\forall A, B \subseteq X)(A \cup B \in u \iff (A \in u \vee B \in u))$,
3. $(\forall A \subseteq X)(A \in u \iff X \setminus A \notin u)$.

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Given a set X, thought of as a discrete topological space, the Čech–Stone compactification of X can be realized as the set βX of all ultrafilters over X, topologized by letting the sets

$$\{u \in \beta X \big| u \in A\}$$

be open, for all $A \subseteq X$.

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An ultrafilter $u \in \beta\omega \setminus \omega$ is said to be **selective** if for every colouring $c : [\omega]^2 \longrightarrow 2$ there exists an $A \in u$ such that $[A]^2$ is c-monochromatic.

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Properties

1. *Selective ultrafilters are minimal in the Rudin-Keisler ordering,*
2. u *is selective iff* $\prod \omega/u$ *has only one constellation (i.e. for any two nonstandard natural numbers* N, M*, there exists a standard* f *such that* $f(N) = M$*).*

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Properties

1. *They are idempotent (i.e. $u + u = u$).*
2. *They have the trivial sums property (that is, whenever $u = v + w$, there must be an $x \in G$ such that $\{v, w\} = \{x + u, -x + u\}$).*

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Theorem (Milliken–Taylor)

For every colouring $c : [\mathbb{B}]^2 \longrightarrow 2$, there exists an infinite $X \subseteq \mathbb{B}$ such that the set

$$[\mathrm{FS}(X)]^2 = \{\langle x, y\rangle \big| x, y \in \mathrm{FS}(X) \qquad \}$$

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$$[\mathrm{FS}(X)]^2_< = \{\langle x, y\rangle \big| x, y \in \mathrm{FS}(X) \text{ and } (\max(x) < \min(y) \vee \max(y) < \min(x))\}$$

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Definition

An ultrafilter $u \in \beta\mathbb{B}$ is said to be **stable ordered union** if for every colouring $c : [\mathbb{B}]^2 \longrightarrow 2$ there exists an infinite ordered $X \subseteq \mathbb{B}$ such that $\mathrm{FS}(X) \in u$ and $[\mathrm{FS}(X)]^2_<$ is monochromatic.

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Questions

1. *Does the existence of a strongly summable ultrafilter imply the existence of a stable ordered union ultrafilter?*
2. *Does the existence of a strongly summable ultrafilter imply the existence of a selective ultrafilter?*

Theme 2: Cardinal Characteristics of the Continuum

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Studying cardinal characteristics of the continuum is, in a sense, a way (the only way that nowadays –after Gödel's and Cohen's results– makes sense) of studying the Continuum Hypothesis, by investigating all of the complexity that might inhabit the space between ω_1 and $\mathfrak{c}$, should the CH fail.

A characteristic associated to Ramsey's theorem

Definition

$$\mathfrak{hom} = \min\{|\mathscr{X}| \,\big|\, (\forall c : [\omega]^2 \longrightarrow 2)(\exists X \in \mathscr{X})([X]^2 \text{ is monochromatic})\}.$$

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(where $\mathfrak{d}$ is the dominating number, and $\mathfrak{r}_\sigma$ is the σ-version of the reaping number).

Characteristics associated to Hindman's/Milliken–Taylor's theorems

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We define $\mathfrak{hom}_H^n$ to be the least cardinality of a family $\mathscr{X}$, each of whose elements is an infinite ordered $X \subseteq \mathbb{B}$, such that for every $c : [\mathbb{B}]^n \longrightarrow 2$ there exists an $X \in \mathscr{X}$ such that $[\mathrm{FS}(X)]^n_<$ is monochromatic.

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It is straightforward to show that we must have $\mathfrak{hom}_H^1 \leq \mathfrak{hom}_H^2 \leq \cdots \leq \mathfrak{hom}_H^n \leq \mathfrak{hom}_H^{n+1} \leq \cdots$. Also, it is known that $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{hom}_H^1$.

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Theorem (F.-B.)

$$\mathfrak{hom}_H^2 = \mathfrak{hom}_H^3 = \cdots = \mathfrak{hom}_H^n = \cdots$$

The unknown

Therefore, there are fundamentally only two cardinal characteristics: $\mathfrak{hom}_H^1$ and $\mathfrak{hom}_H^2$ (let's rename them $\mathfrak{hom}_H$ and $\mathfrak{hom}_{MT}$, respectively). The known relationships are as follows (an arrow means a ZFC-provable inequality).

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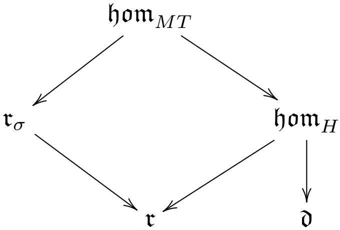

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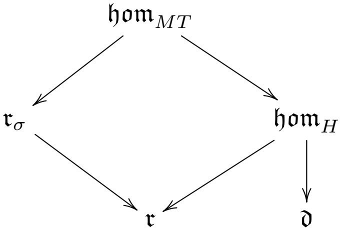

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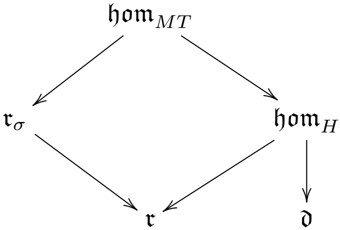

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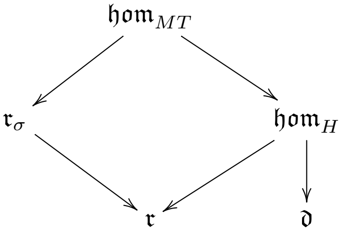
Some open questions



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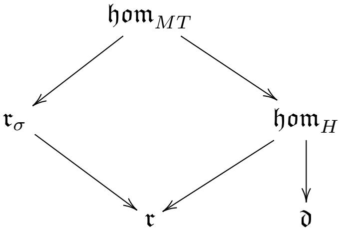

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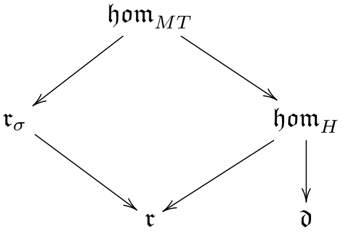

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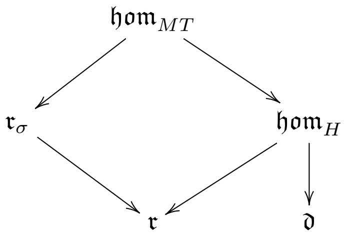

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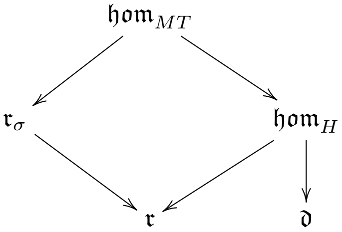

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Theme 3: Uncountable cardinalities

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Theorem (Erdős–Rado)

For every infinite cardinal κ, there exists a sufficiently large λ (in fact, it suffices to take $\lambda = (2^\kappa)^+$) such that for every colouring $c : [\lambda]^2 \longrightarrow 2$ there exists an $X \subseteq \lambda$ with $|X| = \kappa$ such that $[X]^2$ is monochromatic.

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Theorem

*If an uncountable cardinal κ has the property that for every colouring $c : [\kappa]^2 \longrightarrow 2$ there exists an $X \subseteq \kappa$ with $|X| = \kappa$ and $[X]^2$ monochromatic, then κ is very, very large (or actually, not so large... technically, κ is said to be a **weakly compact cardinal**).*

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Theorem (F.-B.)

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Theorem (F.-B. and Rinot)

Let G be any uncountable abelian group. Then there exists a colouring $c : G \longrightarrow \omega$ such that for every uncountable $X \subseteq G$, the set $\mathrm{FS}(X)$ is ***panchromatic****.*

How badly does the uncountable version of Hindman's theorem fail?

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2. *Modulo large cardinals –extremely mild ones–, it is consistent with* ZFC *that for every colouring $c : \mathbb{R} \longrightarrow \omega_1$, there is an uncountable $X \subseteq G$ such that $\mathrm{FS}(X)$ only hits countably many colours.*

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(It is consistent that these κ include all regular cardinals, and it is consistent that $\mathfrak{c}$ finds itself amongst these κ.)

Many colours, small monochromatic sets

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Theorem (Komjáth)

Given any cardinal κ and any $n \in \mathbb{N}$, there exists a sufficiently large λ such that for every colouring $c : \mathbb{B}(\lambda) \longrightarrow \kappa$ there are distinct $x_1, \dots, x_n \in \mathbb{B}(\lambda)$ such that $\mathrm{FS}(x_1, \dots, x_n)$ is monochromatic.

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2. *The "$n = 2$" in our item (1) above is also optimal. That is, there are arbitrarily large abelian groups G such that there exists a $c : G \longrightarrow \omega$ satisfying that for every $x, y, z \in G$, the set*

$$\mathrm{FS}(x, y, z) = \{x, y, z, x + y, y + z, x + z, x + y + z\}$$

is not monochromatic.

Theme 4: Set theory without the Axiom of Choice

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In ZF, it is possible to thoroughly study the sheer variety of different infinite Dedekind-finite sets that might exist. There is a notion of a **finiteness class**. The smallest finiteness class is the class of all finite sets, and the largest finiteness class is the class of all Dedekind-finite sets.

Ramsey's and Hindman's theorems without choice

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Finiteness classes arising from Hindman's theorem

Definition

A set X will be said to be H**-infinite** if for every colouring $c : [X]^{<\omega} \longrightarrow 2$, there exists an infinite $Y \subseteq [X]^{<\omega}$ such that $\mathrm{FS}(X)$ is monochromatic.

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For any set X, the following are equivalent:

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Therefore, most of these notions of finiteness collapse and we are only left with (at most) three of them: H-finite, H_2-finite and H_3-finite.

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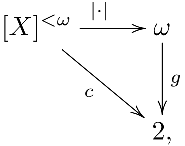

there exists an infinite $Y \subseteq [X]^{<\omega}$ *such that* $\mathrm{FS}_3(Y)$ *is monochromatic.*

Finiteness classes arising from Ramsey's theorem

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Furthermore, none of these arrows is reversible (and similar results where we consider colourings with different numbers of colours).

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Therefore, we now have an instance where Ramsey's theorem implies (a weak version of) Hindman's theorem. In fact, this is just the fact that Ramsey's theorem implies Schur's theorem (i.e. Hindman's for $n = 2$).

Connections with the old notions of finiteness