Algebraic Ramsey-Theoretic Statements with an Uncountable Flavour

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(various joint works with elements of the set $\{\varnothing$, Assaf Rinot, 이성협$\}$)

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Ramsey theoretic statements are always of the form “however you colour a sufficiently large structure, there will always be monochromatic substructures of some prescribed size”.

Theorem (Schur, 1912)

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For every finite colouring of $\mathbb{N}$ and every $k < \omega$ there are two elements a, b such that the set $\{a, a + b, a + 2b, \ldots, a + kb\}$ is monochromatic.

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Theorem (Hindman, 1974)

For every finite colouring of $\mathbb{N}$ *there exists an infinite set* $X \subseteq \mathbb{N}$ *such that the set*

$$\mathrm{FS}(X) = \{x_1 + \cdots + x_n \big| n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in X \text{ are distinct}\}$$

(the set of finite sums of elements of X*) is monochromatic.*

Definition

Let S be a commutative semigroup and let θ, λ be two cardinal numbers. The symbol $S \to (\lambda)_\theta^{\mathrm{FS}}$ will be used to denote the following statement: Whenever we colour the semigroup S with θ colours, there will be a set $X \subseteq S$ with $|X| = \lambda$ such that $\mathrm{FS}(X)$ is monochromatic.

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Thus Hindman's 1974 theorem from the previous slide simply asserts that $\mathbb{N} \to (\aleph_0)_n^{\mathrm{FS}}$ for every finite n. In fact, utilizing the tools from algebra in the Čech–Stone compactification one can prove the following.

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Theorem (Galvin–Glazer–Hindman)

Let G be any infinite abelian group. Then $G \to (\aleph_0)^{\mathrm{FS}}_{n}$ for every finite n.

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Proposition

If G is any infinite abelian group, then $G \nrightarrow (\aleph_0)^{\mathrm{FS}}_{\aleph_0}$.

Theorem (F.B., 2015)

Let G be any uncountable abelian group. Then $G \nrightarrow (\aleph_1)_2^{\mathrm{FS}}$.

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Let G be any uncountable abelian group. Then $G \nrightarrow (\aleph_1)^{\mathrm{FS}}_2$.

Definition

Once again, S is a commutative semigroup and θ, λ are cardinals. The symbol $S \rightarrow (\lambda)^{\mathrm{FS}}_\theta$ denotes the statement that whenever we colour S with θ colours, there will be a set $X \subseteq S$ with $|X| = \lambda$ such that $\mathrm{FS}(X)$ is monochromatic.

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Thus,

Theorem (F.B., 2015)

Let G be any uncountable abelian group. Then $G \nrightarrow [\aleph_1]^{\mathrm{FS}}_2$.

Theorem (Milliken, 1978)

Suppose that G is a group such that $|G| = \kappa^+ = 2^\kappa$ for some cardinal κ. Then $G \nrightarrow [\kappa^+]^{\mathrm{FS}_2}_{\kappa^+}$

(Where $\mathrm{FS}_n(X) = \{x_1 + \cdots + x_n \big| x_1, \ldots, x_n \in X$ are distinct$\}$, so that $\mathrm{FS}(X) = \bigcup_{n\in\mathbb{N}} \mathrm{FS}_n(X)$.)

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Theorem (Hindman, Leader and Strauss, 2015)

For every $n \geq 2$,

$$\mathbb{R} \nrightarrow [\mathfrak{c}]^{\mathrm{FS}_n}_2$$

Theorem (Komjáth and independently D. Soukup and W. Weiss)

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Remark (D. Soukup and W. Weiss)

By a theorem of Shelah, it is consistent with ZFC *(modulo a large cardinal hypothesis) that* $\mathbb{R} \nrightarrow [\omega_1]_3^{\mathrm{FS}_n}$ *fails for every* $n \geq 2$.

Theorem (F.B. and Rinot, 2016)

Let G be any (uncountable) abelian group. Then $G \nrightarrow [\omega_1]^{\mathrm{FS}}_{\omega}$.

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It is consistent with ZFC *(by assuming $\mathbf{V} = \mathbf{L}$ plus the nonexistence of inaccessible cardinals) that $G \nrightarrow [\omega_1]^{\mathrm{FS}}_{\omega_1}$ holds for every uncountable abelian group G.*

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Theorem (F.B. and Rinot, 2016)

Modulo a large cardinal hypothesis (more specifically, the existence of an ω_1-Erdős cardinal), it is consistent with ZFC *that $\mathbb{R} \nrightarrow [\omega_1]^{\mathrm{FS}}_{\omega_1}$ fails.*

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If G is an abelian group of cardinality $\beth_\omega$, then $G \rightarrow [|G|]^{\mathrm{FS}_n}_{\omega}$ (in particular, $G \rightarrow [\omega_1]^{\mathrm{FS}_n}_{\omega}$) for all $n \in \mathbb{N}$.

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Theorem (F.B. and Rinot, 2016)

For every integer $n \geq 2$,

$$\mathbb{R} \nrightarrow [\mathfrak{c}]^{\mathrm{FS}_n}_{\omega},$$

in particular it is consistent (e.g. assuming CH*) that $\mathbb{R} \nrightarrow [\omega_1]^{\mathrm{FS}_n}_{\omega}$.*

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Theorem (F.B. and Rinot, 2016)

Modulo a large cardinal hypothesis (concretely, the existence of a weakly compact cardinal), it is consistent with ZFC *that $\mathbb{R} \nrightarrow [\mathfrak{c}]^{\mathrm{FS}_n}_{\omega_1}$ fails for every integer $n \geq 2$.*

Theorem (F.B. and Rinot, 2016)

The class of cardinals κ for which every abelian group G of cardinality κ satisfies $G \nrightarrow [\kappa]^{\mathrm{FS}_n}_\kappa$ for all $n \geq 2$, includes:

- $\kappa = \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots$; *in fact, every successor of a regular cardinal,*
- *every κ such that $\kappa = \lambda^+ = 2^\lambda$,*
- *every regular uncountable κ admitting a nonreflecting stationary set,*
- *consistently with* ZFC*, every regular uncountable cardinal κ.*

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Given any cardinal κ and any $n \in \mathbb{N}$, there exists a sufficiently large λ (in fact, it suffices to take $\lambda = (\beth_{2^{n-1}-1}(\kappa))^+$) such that $\mathbb{B}(\lambda) \rightarrow (n)_\kappa^{\mathrm{FS}}$.
(Here $\mathbb{B}(\lambda)$ denotes the unique (up to isomorphism) Boolean group of cardinality λ, whose most friendly incarnation is $([\lambda]^{<\omega}, \triangle)$.)

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Given any κ and any $n \in \mathbb{N}$, there exists a sufficiently large cardinal λ (in fact, we can take $\lambda = (\beth_{2^{n-1}}(\beth_{2^{n-1}-1}(\kappa)^+))^+$) such that, for any colouring of $\mathbb{B}(\lambda)$ with κ colours, we can find elements $x_{\alpha,i}$ ($\alpha < \kappa, i < n$) such that the set

$$\{x_{\alpha_0,i_0} + \cdots + x_{\alpha_k,i_k} \big| \alpha_1, \ldots, \alpha_k < \kappa \wedge i_0 < i_1 < \cdots < i_k < n\}$$

is monochromatic. We denote this property with the symbol $\mathbb{B}(\lambda) \rightarrow (\kappa \times n)^{\mathrm{FS}_{\mathrm{matrix}}}_{\kappa}$.

Theorem (Carlucci, 2017)

Given an infinite cardinal κ and positive integers c, d, there exists a λ such that, for every abelian group G of cardinality λ, it is the case that for every c-colouring of G there exists $H \subseteq G$ with $|H| = \kappa$ and $a, b \in \mathbb{N}$ such that the set

$$\bigcup_{n \in \{a, a+b, a+2b, \dots, a+db\}} \mathrm{FS}_n(H)$$

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$$\bigcup_{n \in \mathrm{FS}(\{a_1, \dots, a_d\})} \mathrm{FS}_n(H)$$

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Theorem (F.B. and Lee, 2017)

Given κ, let $\lambda = \beth_1(\kappa)^+ = (2^\kappa)^+$. Then for every abelian group G of cardinality λ, it is the case that

$$G \to (2)^{\mathrm{FS}}_\kappa.$$

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Theorem (F.B. and Lee, 2017)

Given κ, let $\lambda = (2^\kappa)^+$. Then for every abelian group G of cardinality λ, it is the case that

$$G \to (\kappa \times 2)^{\mathrm{FS}_{\mathrm{matrix}}}_\kappa.$$

Definition

An n**-adequate pattern** is a sequence of n elements $\langle x_1, \dots, x_n \rangle \in \bigoplus \mathbb{Z}$ such that for some fixed finite sequence s of nonzero integers, it is the case that

$$\mathrm{NZ}[\mathrm{FS}(\{x_1, \dots, x_n\})] = \{s\},$$

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Proposition (F.B. and Lee, 2017)

The following are equivalent:

- *There exists an n-adequate pattern,*
- *for every κ there exists a λ such that every abelian group G with $|G| = \lambda$ satisfies $G \to (n)^{\mathrm{FS}}_{\kappa}$.*

Definition

We will use the symbol $G \nrightarrow (\lambda)^+_\theta$ to denote the statement that there exists a colouring $c : G \longrightarrow \theta$ such that for every $X \subseteq G$ satisfying $|X| = \lambda$, the set $X + X$ cannot be monochromatic.

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All of the FS_n results of myself and Rinot mentioned previously still hold if we replace FS_2 with $+$ (because $X + X = \mathrm{FS}_2(X) \cup 2X$).

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Question (Owings, 1974)

Is it the case that $\mathbb{N} \nrightarrow (\omega)^+_2$*?*

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We will use the symbol $G \not\rightarrow (\lambda)^+_\theta$ to denote the statement that there exists a colouring $c : G \longrightarrow \theta$ such that for every $X \subseteq G$ satisfying $|X| = \lambda$, the set $X + X$ cannot be monochromatic.

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Theorem (Hindman, 1979)

$\mathbb{N} \not\rightarrow (\omega)^+_3$.

Theorem (Hindman, Leader and Strauss, 2015)

It is consistent with the ZFC *axioms (more concretely, it follows from* $\mathfrak{c} < \aleph_\omega$*) that* $\mathbb{R} \not\rightarrow (\omega)^+_k$ *for some finite* k.

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Theorem (Komjáth, Leader, Russell, Shelah, D. Soukup and Vidnyánszky, 2017)

Modulo large cardinals (more concretely, assuming the existence of a measurable cardinal), it is consistent that $\mathbb{R} \to (\omega)^+_r$ for all finite r.