Set Theory (sometimes) can solve your problem!

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August 31, 2015

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So, the CH is **undecidable** from the ZFC axioms.

Definition

A set $X \subseteq \mathbb{R}$ is said to have **strong measure zero** if for every sequence $\langle \varepsilon_n \big| n \in \mathbb{N} \rangle$ of positive real numbers, there is a sequence of intervals $\langle I_n \big| n \in \mathbb{N} \rangle$ with $\ell(I_n) < \varepsilon_n$ such that

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Assuming CH, *there exists an uncountable set with strong measure zero. So* $\neg$BC *is consistent with* ZFC.

Theorem (Laver 1976)

BC *is consistent with* ZFC. *Hence,* BC *is undecidable from the* ZFC *axioms.*

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Theorem (Shelah 1974)

The ***axiom of constructibility*** *implies that every W-group is free abelian. On the other hand,* ***Martin's axiom*** *implies that there are W-groups of cardinality* $\aleph_1$ *that are not free abelian. In other words, the Whitehead problem is undecidable from the* ZFC *axioms.*

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Theorem (Farah 2011)

The Proper Forcing Axiom/Open Colouring Axiom/Todorcevic's Axiom implies that all automorphisms of the Calkin algebra are inner.