Strongly Summable Ultrafilters in Forcing Extensions

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The Čech-Stone compactification of a discrete abelian group $(G,+)$ is the set βG of all ultrafilters on G, with basic open sets of the form

$$\bar{A} = \{p \in \beta G \big| A \in p\} \qquad (A \subseteq G).$$

Every $x \in G$ is identified with

$$\{A \subseteq G \big| x \in A\},$$

and the group operation $+$ on G is extended by the formula

$$p + q = \{A \subseteq G \big| \{x \in G \big| A - x \in q\} \in p\},$$

making βG into a right-topological compact semigroup with $G^* = \beta G \setminus G$ as a closed subsemigroup.

Let X be a subset (typically infinite) of elements of G.

$$\mathrm{FS}(X) = \left\{ \sum_{x \in F} x \middle| F \in [X]^{<\omega} \setminus \{\varnothing\} \right\}.$$

Definition

We say that $p \in G^*$ is **strongly summable** if for every $A \in p$ there exists an infinite $X \subseteq G$ such that $p \ni \mathrm{FS}(X) \subseteq A$ (i.e. p has a basis of FS-sets).

Theorem (Hindman-Blass on $\mathbb{Z}$, Hindman-Protasov-Strauss in general)

Every strongly summable ultrafilter p is an idempotent (i.e. $p = p + p$).

Theorem (Hindman-Protasov-Strauss on $\mathbb{T}$, F.B. in general)

If G is an abelian group and $p \in G^*$ is strongly summable, then whenever $q, r \in G^*$ are such that $q + r = p$, it must be the case that $q, r \in G + p$.

"The Boolean Group" will be $([\omega]^{<\omega}, \triangle)$. We will denote it by $\mathbb{B}$.

Theorem (Blass-Hindman, Hindman-Steprāns-Strauss, F.B.)

Given an arbitrary abelian group G, every strongly summable ultrafilter on G "looks like" a strongly summable ultrafilter on $\mathbb{B}$.

The cardinal invariant $\mathrm{cov}(\mathcal{M})$ is the least number of meagre sets needed to cover $\mathbb{R}$. It happens to equal $\mathfrak{m}(\mathrm{countable})$, hence writing that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ means that Martin's Axiom holds when restricted to countable partial orders.

Theorem (Hindman, Blass-Hindman, Eisworth)

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ *then there exists a strongly summable ultrafilter on* $\mathbb{B}$. *Moreover, whenever we have* $< \mathfrak{c}$ *many sets* $X_\alpha \subseteq \mathbb{B}$, $\alpha < \mathfrak{c}$ *such that* $\mathscr{F} = \{\mathrm{FS}(X_\alpha \setminus Z) \big| \alpha < \mathfrak{c} \wedge X \in [X_\alpha]^{<\omega}\}$ *has the strong finite intersection property, then there exists a strongly summable ultrafilter extending* $\mathscr{F}$*.*

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Theorem (Hindman, Blass-Hindman, Eisworth)

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ then there exists a strongly summable ultrafilter on $\mathbb{B}$. Moreover, whenever we have $< \mathfrak{c}$ many sets $X_\alpha \subseteq \mathbb{B}$, $\alpha < \mathfrak{c}$ such that $\mathscr{F} = \{\mathrm{FS}(X_\alpha \setminus Z) \big| \alpha < \mathfrak{c} \wedge X \in [X_\alpha]^{<\omega}\}$ has the strong finite intersection property, then there exists a strongly summable ultrafilter extending $\mathscr{F}$.

Question

Is it consistent that $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$ and there exists a strongly summable ultrafilter on $\mathbb{B}$?

Definition

Given an ultrafilter $p \in \mathbb{B}^*$ such that $p + p = p$ (an idempotent ultrafilter), we define $\mathbb{L}(p)$ to be the **ultraLaver forcing**. This is, conditions T are subtrees of $\mathbb{B}^{<\omega}$ that have a *stem* $s(T)$ (i.e. a node such that every $t \in T$ is comparable to $s(T)$) and such that, for every $t \in T$ with $t \geq s(T)$, we have that

$$\{x \in \mathbb{B} \big| t \frown x \in T\} \in p.$$

The ordering is just inclusion: $T \leq T'$ iff $T \subseteq T'$.

If G is a generic filter obtained by forcing with $\mathbb{L}(p)$, we let

$$X = \bigcap G = \bigcup_{T \in G} s(T).$$

Lemma

Every ground model $A \subseteq \mathbb{B}$ is "diagonalized" by X, in the sense that there exists a finite $F \subseteq X$ such that $\mathrm{FS}(X \setminus F)$ is either $\subseteq A$ or disjoint from A, depending on whether or not $A \in p$.

Start with two regular cardinals κ, λ such that $\lambda < \mathfrak{c} = \kappa$ in the ground model. Define an FS iteration $\langle \mathbb{P}_\alpha \big| \alpha \leq \lambda \rangle$ with iterands $\mathring{\mathbb{Q}}_\alpha$, by recursively defining names $\mathring{p}_\alpha, \mathring{\mathbb{Q}}_\alpha, \mathring{X}_\alpha$ (in that order) such that:

$$\begin{aligned} \mathbb{P}_\alpha \quad \Vdash \quad & \text{``}\mathring{p}_\alpha \text{ is an idempotent ultrafilter extending} \\ & \{\mathrm{FS}(\mathring{X}_\xi \setminus F) \big| F \in [\mathring{X}_\xi]^{<\omega} \wedge \xi < \alpha\}\text{''}, \end{aligned}$$

$$\mathbb{P}_\alpha \Vdash \text{``}\mathring{\mathbb{Q}}_\alpha = \mathbb{L}(\mathring{p}_\alpha)\text{''},$$

and $\mathring{X}_\alpha$ is the name for the generic set added to $V^{\mathbb{P}_\alpha}$ by $\mathbb{L}(\mathring{p}_\alpha)$. Forcing with $\mathbb{P}_\lambda$ yields a model with a strongly summable ultrafilter and $\mathrm{cov}(\mathcal{M}) = \lambda < \kappa = \mathfrak{c}$.

Definition

A forcing notion $\mathbb{P}$ satisfies the **Laver property** if whenever $g : \omega \longrightarrow \omega$ (in the ground model) and $\mathring{f}$ is the name of a function $: \omega \longrightarrow \omega$ such that $\Vdash$ "$\mathring{f} \leq \check{g}$", there is $F : \omega \longrightarrow [\omega]^{<\omega}$ such that for every $n < \omega$, $|F(n)| \leq 2^n$ and $\Vdash$ "$\check{g}(\check{n}) \in \check{F}(\check{n})$".

The Laver property is important because (1) it is preserved under CS iterations, and (2) whenever $\mathbb{P}$ has the Laver property, it does not add any Cohen reals. Hence if we force with a CS iteration of forcings satisfying the Laver property, $\mathrm{cov}(\mathcal{M})$ in the generic extension will have the same value that it used to have in the ground model.

Theorem (F.B.)

If p is a stable ordered union ultrafilter, then $\mathbb{L}(p)$ satisfies the Laver property.

Hence we define a CS iteration of length ω_2, by recursively defining names $\mathring{p}_\alpha, \mathring{\mathbb{Q}}_\alpha, \mathring{X}_\alpha$ as follows: $\mathbb{P}_\alpha \Vdash$ "$\mathring{\mathbb{Q}}_\alpha = \mathbb{L}(\mathring{p}_\alpha)$" and $\mathring{X}_\alpha$ is the name for the generic set added to $V^{\mathbb{P}_\alpha}$ by $\mathbb{L}(\mathring{p}_\alpha)$.

The main difficulty lies in defining the $\mathring{p}_\alpha$. We let p_0 be any stable ordered union ultrafilter (note that the ground model, as well as all intermediate extensions, satisfy CH). $\mathring{p}_{\alpha+1}$ is always an ordered union ultrafilter extending $\{\mathrm{FS}(\mathring{X}_\alpha \setminus F) \big| F \in [\mathring{X}_\alpha]^{<\omega}\}$, and for $\alpha = \bigcup \alpha$ of uncountable cofinality we just let $\mathring{p}_\alpha$ be the (ultra)filter generated by $\{\mathrm{FS}(\mathring{X}_\xi \setminus F) \big| F \in [\mathring{X}_\xi]^{<\omega} \wedge \xi < \alpha\}$.

Finally, if $\alpha = \bigcup \alpha$ has countable cofinality, we let $\langle \alpha_n \big| n < \omega \rangle$ be an increasing sequence, cofinal in α, and let $\mathring{p}_\alpha$ be (the name of) any stable ordered union ultrafilter extending $\{\mathrm{FS}(\mathring{X}_{\alpha_n} \setminus F) \big| F \in [\mathring{X}_{\alpha_n}]^{<\omega} \wedge n < \omega\}$.

Forcing with $\mathbb{P}_{\omega_2}$ yields a model with a strongly summable ultrafilter (in fact, a stable ordered union ultrafilter). By the previously mentioned results about the Laver property, in this model $\mathrm{cov}(\mathcal{M}) = \omega_1 < \omega_2 = \mathfrak{c}$.

Model	$\exists$ s.s. u.f.s on $\mathbb{B}$?	Reason
FS ultraLaver	Yes	Ad-hoc
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Shelah's no P-point	No	No P-points
Laver	No	No rapid ultrafilters
Mathias	No	No rapid ultrafilters
Random	No	No rapid P-points
Miller	No	NCF

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Sacks	?	?

Question

Are there strongly summable ultrafilters on $\mathbb{B}$ in Sacks's model?