Some results concerning Strongly Summable Ultrafilters on Abelian Groups

David J. Fernández Bretón

Department of Mathematics and Statistics
York University

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The Čech-Stone compactification of a discrete abelian group $(G, +)$ is the set βG of ultrafilters on G, with basic open sets of the form

$$\bar{A} = \{p \in \beta G \big| A \in p\} \qquad\qquad (A \subseteq G).$$

Every $x \in S$ is identified with

$$\{A \subseteq G \big| x \in A\},$$

and the group operation $+$ on G is extended by the formula

$$p + q = \{A \subseteq G \big| \{x \in G \big| A - x \in q\} \in p\},$$

and $G^* = \beta G \setminus G$ is a closed subsemigroup.

Denote by $\vec{x} = \langle x_n \big| n < \omega \rangle$ a sequence (typically injective) of elements of G.

$$\mathrm{FS}(\vec{x}) = \left\{ \sum_{n \in a} x_n \middle| a \in [\omega]^{<\omega} \setminus \{\varnothing\} \right\}.$$

Definition

We say that $p \in G^*$ is **strongly summable** if for every $A \in p$ there exists a sequence $\vec{x}$ such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A$.
(i.e. p has a basis of FS-sets)

Theorem (Hindman-Blass on $\mathbb{Z}$, Hindman-Protasov-Strauss in general)

Every strongly summable ultrafilter p is an idempotent (i.e. $p = p + p$).

Theorem (Hindman-Strauss)

Let $p \in \mathbb{Z}^*$ be a strongly summable ultrafilter, and let $q, r \in \omega^*$ be such that $q + r = r + q = p$. Then $q, r \in \mathbb{Z} + p$.

Theorem (Hindman-Protasov-Strauss)

If $G \subseteq \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $p \in G^*$ is strongly summable, then whenever $q, r \in G^*$ are such that $q + r = r + q = p$, it must be the case that $q, r \in G + p$.

Definition

We say that $p \in G^*$ has the **trivial sums property** if whenever $q, r \in G^*$ are such that $q + r = p$, we must have that $q, r \in G + p$.

Definition (Hindman-Protasov-Strauss)

An ultrafilter $p \in G^*$ is **sparse** if for every $A \in p$ there exist a sequence $\vec{x} = \langle x_n \big| n < \omega \rangle$ and a moiety M of ω such that $\mathrm{FS}(\vec{x}) \subseteq A$ and $\mathrm{FS}(\langle x_n \big| n \in M \rangle) \in p$.

Theorem (Hindman-Protasov-Strauss)

If $G \subseteq \mathbb{T}$ and $p \in G^*$ is sparse, then p has the trivial sums property.

Theorem (Hindman-Steprāns-Strauss)

We can assume that $G \subseteq \bigoplus_{n<\omega} \mathbb{T}$. If $p \in G^*$ is a strongly summable ultrafilter, and

$$\left\{ x \in S \middle| \pi_{\rho(x)}(x) \neq \frac{1}{2} \right\} \in p, \qquad (\rho(x) = \min\{i < \omega | \pi_i(x) \neq 0\})$$

then p is sparse.

Theorem (F.B.)

No matter what G is, if $p \in G^*$ is strongly summable then it is sparse and it has the trivial sums property.

Definition (Blass)

A a nonprincipal ultrafilter p on $[\omega]^{<\omega}$ is called a **union ultrafilter** if for every $A \in p$ there exists a pairwise disjoint sequence $\vec{x} = \langle x_n \big| n < \omega \rangle$ of elements of $[\omega]^{<\omega}$ such that $p \ni \mathrm{FU}(\vec{x}) \subseteq A$.

Definition (Hindman-Blass)

We will say that a strongly summable ultrafilter $p \in G^*$ is **additively isomorphic to a union ultrafilter** if for some sequence $\vec{x}$ of elements of G with $\mathrm{FS}(\vec{x}) \in p$, the mapping $\sum_{i \in a} x_i \longmapsto a$ sends p to a union ultrafilter.

Theorem (Blass-Hindman for $\mathbb{Z}$, F.B. in general)

If $p \in G^*$ and $\{x \in G \big| 2x = 0\} \notin p$ then p is additively isomorphic to a union ultrafilter.

Theorem (F.B.)

Assume $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. Then, there exists a strongly summable ultrafilter on the Boolean group $([\omega]^{<\omega}, \triangle)$ which is not additively isomorphic to a union ultrafilter.

Theorem (Hindman-Blass for $\mathbb{Z}$, Hindman-Protasov-Strauss in general)

Given G, there exists a mapping $\mu : G \longrightarrow \omega$ such that, if $p \in G^*$ is strongly summable, then $\mu(p)$ is a P-point. Furthermore, if $\{x \in G \big| 2x = 0\} \notin p$, then $\mu(p)$ is rapid.

Theorem (Krautzberger)

If $p \in \mathbb{Z}^*$ then p is rapid. (Hence, if $p \in G^*$ and $\{x \in G \big| 2x = 0\} \notin p$, then p is rapid).

Theorem (F.B.)

Let $\max : [\omega]^{<\omega} \longrightarrow \omega$. If p is a strongly summable ultrafilter on the Boolean group $([\omega]^{<\omega}, \triangle)$, then both $\max(p)$ and p itself are rapid.

If p is a strongly summable ultrafilter on $G = ([\omega]^{<\omega}, \triangle)$, let

$$\Pr(p) = \{(A, \mathrm{FS}(X)) \big| A \in [G]^{<\omega} \wedge \mathrm{F}\triangle(X) \in p\}$$

with $(A, \mathrm{FS}(X)) \leq (B, \mathrm{FS}(Y))$ iff $A \supseteq B$ and $\mathrm{FS}(X \cup (A \setminus B)) \subseteq \mathrm{F}\triangle(Y)$ (equivalently $X \cup (A \setminus B) \subseteq \mathrm{FS}(Y)$).

Theorem (F.B.)

Let $\omega < \lambda < \kappa$ be two regular cardinals. A finite support iteration of length λ, with iterands of the form $\Pr(p) \star \mathrm{Fn}(\kappa, 2)$ ($\mathrm{Fn}(\kappa, 2)$ is the forcing notion that adds κ many Cohen reals) yields a model where $\mathrm{cov}(\mathcal{M}) = \lambda$, $\mathfrak{c} = \kappa$ and there exists a strongly summable ultrafilter on G (actually, this ultrafilter is generated by λ elements, so $\mathfrak{u} = \lambda$ as well).

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