Ends of nonmetrizable manifolds: a generalized bagpipe theorem

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ENDS OF NON-METRIZABLE MANIFOLDS:
A GENERALIZED BAGPIPE THEOREM

DAVID FERNÁNDEZ-BRETÓN AND NICHOLAS G. VLAMIS,
AN APPENDIX WITH MATHIEU BAILLIF

Abstract. We initiate the study of ends of non-metrizable manifolds and introduce the notion of short and long ends. Using the theory developed, we provide a characterization of (non-metrizable) surfaces that can be written as the topological sum of a metrizable manifold plus a countable number of "long pipes" in terms of their spaces of ends; this is a direct generalization of Nyikos's bagpipe theorem.

1. Introduction

An n-manifold is a connected Hausdorff topological space that is locally homeo-

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connected topological space such that, for some fixed $n \in \mathbb{N}$, we have $(\forall x \in M)(\exists U \ni x \text{ open})(M \approx \mathbb{R}^n)$.

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A **manifold** M is a paracompact connected topological space such that, for some fixed $n \in \mathbb{N}$, we have $(\forall x \in M)(\exists U \ni x \text{ open})(M \approx \mathbb{R}^n)$.

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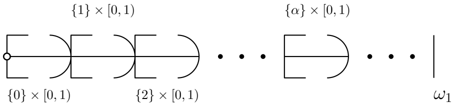

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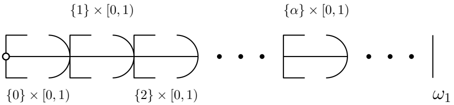

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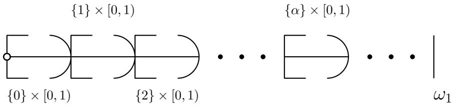

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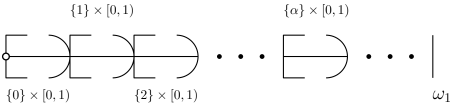

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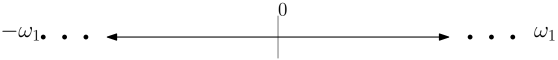

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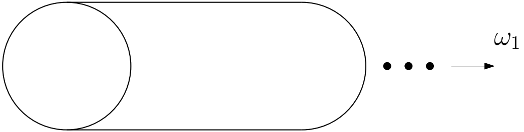

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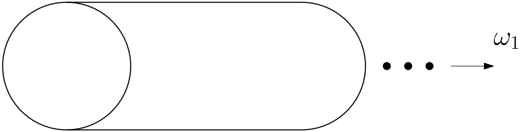

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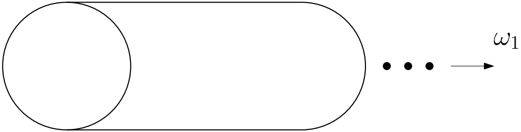

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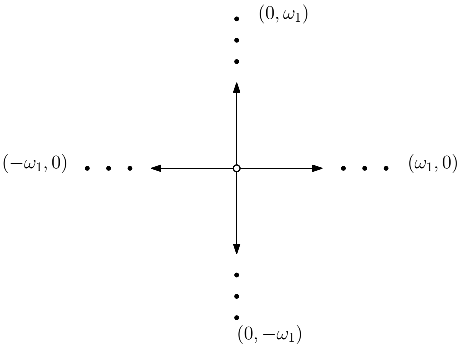

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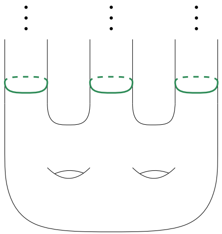

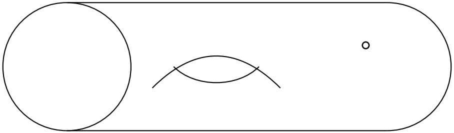

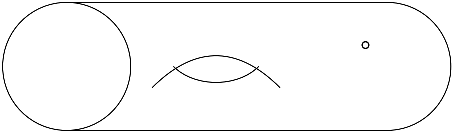

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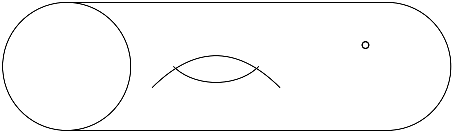
An **end** of a manifold M is an element of the remainder $\mathcal{E}(M) = \mathcal{F}(M) \setminus M$, where $\mathcal{F}(M)$ is the Freudenthal compactification of M.



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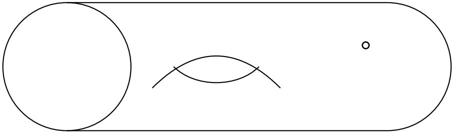

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Theorem (Richards's classification theorem, 1963)

A metrizable 2-manifold M is determined by its genus, orientability class, and end space (plus suitable information on the ends).

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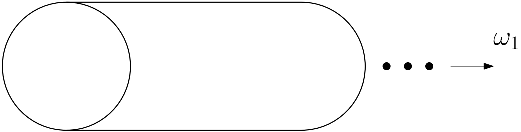

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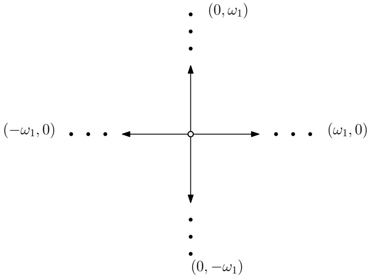

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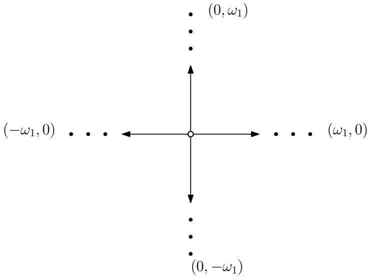

2. $\mathbb{L}^{\leftrightarrow} \times \mathbb{R}$ has one end

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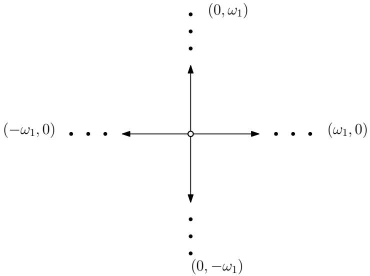

2. $\mathbb{L}^{\leftrightarrow} \times \mathbb{R}$ has one end that is neither short nor long:

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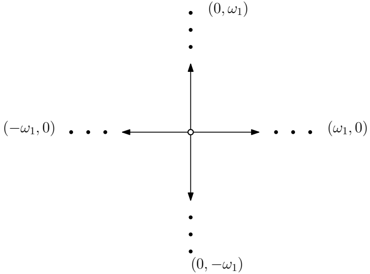

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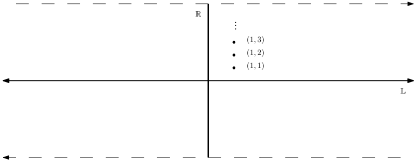

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1. *M is metrizable,*
2. *$\mathcal{E}(M)$ is second countable and every end of M is short.*

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Theorem (Baillif, F.B., Vlamis)

The assumption that M is type I cannot be removed from the above theorem.

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If M is a 2-manifold of type I, then the following are equivalent:

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Theorem (F.B., Vlamis)

If M is a 2-manifold of type I, then the following are equivalent:

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Theorem (Baillif, F.B., Vlamis)

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Friday, 3:00–3:20, Continuum Theory session:

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Equicontinuous mappings on finite trees

by

Gerardo Acosta and **David Fernández-Bretón** (Ciudad de México)

Abstract. If X is a finite tree and $f\colon X \to X$ is a map, in the Main Theorem of this paper (Theorem 1.8), we find eight conditions, each of which is equivalent to f being equicontinuous. To name just a few of the results obtained: the equicontinuity of f is equivalent to there being no arc $A \subseteq X$ satisfying $A \subsetneq f^n[A]$ for some $n \in \mathbb{N}$. It is also equivalent to the statement that for some nonprincipal ultrafilter u, the function $f^u\colon X \to X$ is continuous (in other words, failure of equicontinuity of f is equivalent to the failure of continuity of *every* element of the Ellis remainder $g \in E(X, f)^*$). One of the tools used in the proofs is the Ramsey-theoretic result known as Hindman's theorem.