STRONGLY PRODUCTIVE ULTRAFILTERS ON SEMIGROUPS

DAVID J. FERNÁNDEZ BRETÓN AND MARTINO LUPINI

ABSTRACT. We prove that if S is a commutative semigroup with well-founded universal semilattice or a solvable inverse semigroup with well-founded semilattice of idempotents, then every strongly productive ultrafilter on S is idempotent. Moreover we show that any very strongly productive ultrafilter on the free semigroup with countably many generators is sparse, answering a question of Hindman and Legette Jones.

1. INTRODUCTION

Let S be a multiplicatively denoted semigroup. If $\vec{x} = (x_n)_{n\in\omega}$ is a sequence of elements of S, then the *finite products set* associated with $\vec{x}$, denoted by $\mathrm{FP}(\vec{x})$, is the set of products (taken in increasing order of indices)

$$\prod_{i\in a} x_i \in S$$

where a ranges among the finite subsets of ω. If $k \in \omega$ then $\mathrm{FP}_k(\vec{x})$ stands for the FP-set associated with the sequence $(x_{n+k})_{n\in\omega}$. A subset of S is called an FP-set if it is of the form $\mathrm{FP}(\vec{x})$ for some sequence $\vec{x}$ in S, and an IP-set if it contains an FP-set (see [12, Definition 16.3]). An ultrafilter p on S (a gentle introduction to ultrafilters can be found in [2, Appendix B]) is *strongly productive* as in [9, Section 1]) if it has a basis of FP-sets. This means that for every $A \in p$ there is an FP-set contained in A that is a member of p. When S is an additively denoted commutative semigroup, then the finite products sets are called *finite sums sets* or FS-sets and denoted by $\mathrm{FS}(\vec{x})$. Moreover strongly productive ultrafilters are called in this context *strongly summable* (see [10, Definition 1.1]).

The concept of strongly summable ultrafilter was first considered in the case of the semigroup of positive integers in [8] by Hindman upon suggestion of van Douwen (see also the notes at the end of [12, Chapter 12]). Later Hindman, Protasov, and Strauss studied in [10] strongly summable ultrafilters on arbitrary abelian groups. Theorem 2.3 in [10] asserts that any strongly summable ultrafilter on an abelian group G is idempotent, i.e. an idempotent element of the semigroup compactification βG of G, which can be seen as the collection of all ultrafilters on G. Similarly, strongly productive ultrafilters on a free semigroup are also idempotent by [9, Lemma 2.3].

2010 *Mathematics Subject Classification.* Primary 54D80; Secondary 54D35, 20M14, 20M18, 20M05.

Key words and phrases. Ultrafilters, strongly productive ultrafilters, commutative semigroups, solvable groups, solvable inverse semigroups, Čech-Stone compactification, finite products.

David Fernández was supported by scholarship number 213921/309058 of the Consejo Nacional de Ciencia y Tecnología (CONACyT), Mexico. Martino Lupini was supported by the York University Elia Scholars Program.

In this paper we provide a common generalization of these results to a class of semigroups containing in particular all commutative semigroups with well-founded universal semilattice, and solvable inverse semigroups with well-founded semilattice of idempotents. (The notion of universal semilattice of a semigroup is presented in [6, Section III.2]. Inverse semigroups are introduced in [6, Section II.2], and within these the class of solvable semigroups is defined in [15, Definition 3.2]. Solvable groups are the solvable inverse semigroups with exactly one idempotent element by [15, Theorem 3.4].)

Theorem 1.1. *If S is either a commutative semigroup with well-founded universal semilattice, or a solvable inverse semigroup with well-founded semilattice of idempotents, then every strongly productive ultrafilter on S is idempotent.*

In order to prove Theorem 1.1 we find it convenient to consider the following strengthening of the notion of strongly productive ultrafilter:

Definition 1.2. A nonprincipal strongly productive ultrafilter p on a semigroup S is *regular* if it contains an element B with the following property: Whenever $\vec{x}$ is a sequence in S such that $\mathrm{FP}(\vec{x}) \subset B$, the set $x_0 \, \mathrm{FP}_1(\vec{x})$ does not belong to p.

Remark 1.3. Suppose that S, T are semigroups, $f : S \to T$ is a semigroup homomorphism, and p is a strongly productive ultrafilter on S. Denote by q the ultrafilter on T defined by $B \in q$ if and only if $f^{-1}[B] \in p$ (note that q is the image of p under the unique extension of f to a continuous function from the Čech-Stone compactification of S to the Čech-Stone compactification of T). It is easy to see that q is a strongly productive ultrafilter on T. Moreover if q is regular then p is regular.

We will show that the notions of strongly productive and regular strongly productive ultrafilter coincide for the classes of semigroups considered in Theorem 1.1.

Theorem 1.4. *If S is either a commutative semigroup with well-founded universal semilattice, or a solvable inverse semigroup with well-founded semilattice of idempotents, then every nonprincipal strongly productive ultrafilter on S is regular.*

The *universal semilattice* of a semigroup S is the quotient of S by the smallest semilattice congruence $\mathcal{N}$ on S, see [6, Section III.2]. When S is a commutative inverse semigroup the set $E(S)$ of idempotent elements of S is a semilattice, and the restriction to $E(S)$ of the quotient map from S to $S/\mathcal{N}$ is an isomorphism from $E(S)$ onto $S/\mathcal{N}$. Recall also that an ordered set is said to be *well-founded* if every nonempty subset has a least element.

Although not using this terminology, there is a well-known argument showing that any regular strongly productive ultrafilter is idempotent, see for example [12, Theorem 12.19], or [9, Lemma 2.3]. We reproduce the argument in Lemma 1.5 below for convenience of the reader. Using this fact, Theorem 1.1 will be a direct consequence of Theorem 1.4.

Lemma 1.5. *Suppose that S is a semigroup, and p is an ultrafilter on S. If p is regular strongly productive, then p is idempotent.*

Proof. Fix an element B of p witnessing the fact that p is regular. Suppose that A is an element of p and $\vec{x}$ is a sequence in S such that $\mathrm{FP}(\vec{x}) \subset A \cap B$. Since

$$\mathrm{FP}(\vec{x}) = \{x_0\} \cup x_0 \, \mathrm{FP}_1(\vec{x}) \cup \mathrm{FP}_1(\vec{x}),$$

and p is nonprincipal and regular, it follows that $\mathrm{FP}_1(\vec{x}) \in p$. Using this argument, one can show by induction that $\mathrm{FP}_n(\vec{x}) \in p$ for every $n \in \omega$. Now notice that, if $x = \prod_{i \in a} x_i \in \mathrm{FP}(\vec{x})$ and $n = \max(a) + 1$, then $x \, \mathrm{FP}_n(\vec{x}) \subseteq \mathrm{FP}(\vec{x}) \subseteq A$. Therefore $\mathrm{FP}_n(\vec{x}) \subseteq x^{-1}A$ and since the former set is an element of p, so is the latter. Hence

$$\mathrm{FP}(\vec{x}) \subset \left\{ x \in S : x^{-1}A \in p \right\}$$

and so the latter set belongs to p. This shows that p is idempotent. □

To our knowledge it is currently not known if the existence of a semigroup S and a nonprincipal strongly productive ultrafilter on S that is not idempotent is consistent with the usual axioms of set theory. We think that Theorem 1.1 as well as [10, Theorem 2.3] provide evidence that for all semigroups S, every strongly productive ultrafilter on S should be idempotent.

Conjecture 1.6. *Every strongly productive ultrafilter on an arbitrary semigroup is idempotent.*

This paper is organized as follows: In Section 2 we introduce the notion of IP-regular (partial) semigroup and observe that IP-regular semigroups satisfy the conclusion of Theorem 1.4. In Section 3 we show that commutative cancellative semigroups are IP-regular. In Section 4 we record some closure properties of the class of IP-regular semigroups, implying in particular that all (virtually) solvable groups are IP-regular. In Section 5 we present the proof of Theorem 1.4. Finally in Section 6 we discuss sparseness of strongly productive ultrafilters, and show that every very strongly productive ultrafilter on the free semigroup with countably many generators is sparse, answering Question 3.8 from [9].

Acknowledgments. We would like to thanks the anonymous referee and Jimmie Lawson for their comments and remarks.

2. IP-regularity

A *partial semigroup* as defined in [7, Section I.3] is a set P endowed with a partially defined (multiplicatively denoted) operation such that for every $a, b, c \in P$

$$(ab)c = a(bc)$$

whenever both $(ab)c$ and $a(bc)$ are defined. Observe that in particular every semigroup is a partial semigroup. Moreover any subset of a partial semigroup is naturally endowed with a partial semigroup structure when one considers the restriction of the operation. An element a of a partial semigroup P is idempotent if $a \cdot a$ is defined and equal to a. The set of idempotent elements of P is denoted by $E(P)$. The notion of FP-set and IP-set admit straightforward generalizations to the framework of partial semigroups. If $\vec{x}$ is a sequence of elements of a partial semigroup P such that all the products (taken in increasing order of indices)

$$\prod_{i \in a} x_i$$

where a is a finite subset of ω are defined, then the FP-set $\mathrm{FP}(\vec{x})$ is the set of all such products. A subset of P is an IP-set if it contains an FP-set. Analogous to the case of semigroups, an ultrafilter p on a partial semigroup P is *strongly productive* if it is has a basis of FP-sets. A strongly productive ultrafilter is *regular* if it contains a set B with the following property: If $\vec{x}$ is a sequence in P such that all finite products from $\vec{x}$ are defined and belong to B, then $x_0 \operatorname{FP}_1(\vec{x})$ does not belong to p.

Definition 2.1. A partial semigroup P is *strongly* IP*-regular* if for every sequence $\vec{x}$ in P such that all the finite products from $\vec{x}$ are defined the set

$$x_0 \operatorname{FP}_1(\vec{x})$$

is not an IP-set.

Since a subset of a partial semigroup is still a partial semigroup, we can speak about strongly IP-regular subsets of a partial semigroups, which are just strongly IP-regular partial semigroups with respect to the induced partial semigroup structure.

Definition 2.2. A partial semigroup P is IP*-regular* if $P \setminus E(P)$ is the union of finitely many strongly IP-regular sets.

It is immediate from the definition that a partial semigroup is IP-regular whenever it is the union of finitely many IP-regular subsets (i.e. subsets which are IP-regular partial semigroups with respect to the induced partial semigroup structure).

Remark 2.3. Any subset of a (strongly) IP-regular partial semigroup is (strongly) IP-regular. Any finite partial semigroup is strongly IP-regular.

We will show in Section 3 that every cancellative commutative semigroup is IP-regular. The relevance of the notion of IP-regularity stems from the fact that any IP-regular group satisfies the conclusion of Theorem 1.4. The same is in fact true for any IP-regular partial semigroup with finitely many idempotent elements.

Lemma 2.4. *If S is an* IP*-regular partial semigroup with finitely many idempotent elements, then every nonprincipal strongly productive ultrafilter on S is regular.*

Proof. Suppose that p is a strongly summable ultrafilter on S. Since p is nonprincipal, $S \setminus E(S) \in p$. Given that S is IP-regular, the ultrafilter p must have a strongly IP-regular member A. It is clear that A witnesses the fact that p is regular. □

The class of IP-regular partial semigroups has interesting closure properties. We have already observed that a subset of an IP-regular partial semigroup is IP-regular. Moreover the inverse image of an IP-regular partial semigroup under a partial homomorphism is IP-regular. Recall that a *partial homomorphism* from a partial semigroup P to a partial semigroup Q is a function $f : P \to Q$ such that for every $a, b \in P$ such that ab is defined, $f(a)f(b)$ is defined and $f(ab) = f(a)f(b)$

Lemma 2.5. *Suppose that $f : P \to Q$ is a partial homomorphism. If Q is* IP*-regular and $f^{-1}[E(Q)]$ is* IP*-regular, then P is* IP*-regular.*

Proof. Observe that the image of an IP-set under a partial homomorphism is an IP-set. It follows that $f^{-1}[B]$ is strongly IP-regular whenever $B \subset Q \setminus E(Q)$ is strongly IP-regular. The fact that P is IP-regular follows easily from these observations. □

In particular Lemma 2.5 guarantees that the extension of an IP-regular group by an IP-regular group is IP-regular. More generally one can consider groups admitting a subnormal series with IP-regular factor groups. Recall that a subnormal series of a group G is a finite sequence (a) of subgroups of G such that $A_0 = \{1_G\}$, $A_n = G$, and $A_i \trianglelefteq A_{i+1}$ for every $i \in n$. The quotients A_{i+1}/A_i for $i \in n$ are called *factor groups* of the series. Proposition 2.6 can be easily obtained from Lemma 2.5 by induction on the length of the subnormal series.

Proposition 2.6. *Suppose that G is a group, and $(A_i)_{i\in n}$ is a subnormal series of G. If the factor groups A_{i+1}/A_i are* IP*-regular for all $i \in n$, then G is* IP*-regular.*

A consequence of Proposition 2.6 is that solvable groups are IP-regular. This follows from the facts that solvable groups are exactly those that admit a subnormal series where all the factor groups are abelian, and that all abelian groups are IP-regular. The latter fact will be proved in the following section.

3. Cancellative commutative semigroups

Throughout this section all (partial) semigroups are *additively denoted* and assumed to be *cancellative* and *commutative.*

Proposition 3.1. *Every cancellative commutative semigroup is* IP*-regular (as in Definition 2.2).*

A key role in the proof of Proposition 3.1 is played by the notion of rank function.

Definition 3.2. A *rank function* on a cancellative commutative partial semigroup P is a function ρ from P to a well-ordered set with the property that if $\vec{x}$ is a sequence in P such that all finite sums from $\vec{x}$ are defined, then the two following conditions are satisfied:

(1) The restriction of ρ to the range $\{x_n \mid n \in \omega\}$ of the sequence $\vec{x}$ is a finite-to-one function, and
(2) if $\rho(x_n) \geq \rho(x_0)$ for every $n \in \omega$ then $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set.

Remark 3.3 provides an example of a rank function.

Remark 3.3. If ρ is a function from P to a well order such that $\rho(x+y) = \min\{\rho(x), \rho(y)\}$ and $\rho(x) \neq \rho(y)$ whenever $x, y \in P$ and $x + y$ is defined, then ρ is a rank function on A.

The relevance of rank functions for the proof of Proposition 3.1 is stated in Lemma 3.4. Recall that the family of IP-sets of a (partial) semigroup P is partition regular (see [12, Corollary 5.15]). This means that if $\mathfrak{F}$ is a finite family of subsets of P and $\bigcup\mathfrak{F}$ is an IP-set, then $\mathfrak{F}$ contains an IP-set. This fact will be used in the proof of Lemmas 3.4 and 3.6

Lemma 3.4. *If there is a rank function on a partial semigroup P, then P is strongly* IP*-regular.*

Proof. Fix a rank function ρ from P to a well-ordered set. Suppose that $\vec{x}$ is a sequence in P such that all the finite sums are defined. We claim that $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set. Since ρ restricted to the range of $\vec{x}$ is finite-to-one, it is possible to pick a permutation σ of ω such that

$$\rho\left(x_{\sigma(n)}\right) \leq \rho\left(x_{\sigma(m)}\right)$$

for every $n \in m \in \omega$. Define $y_n = x_{\sigma(n)}$ and $\vec{y} = (y_n)_{n\in\omega}$. Observe that if $x_0 = y_{n_0}$ then

$$\begin{aligned} & x_0 + \mathrm{FS}_1(\vec{x}) \\ = \quad & (y_{n_0} + \mathrm{FS}_{n_0+1}(\vec{y})) \cup \left(y_{n_0} + \mathrm{FS}\left((y_i)_{i=0}^{n_0-1} \right) \right) \cup \\ & \cup \left(y_{n_0} + \mathrm{FS}\left((y_i)_{i=0}^{n_0-1} \right) + \mathrm{FS}_{n_0+1}(\vec{y}) \right). \end{aligned}$$

Applying the hypothesis that ρ is a rank function to the sequence $(y_{n_0+k})_{k\in\omega}$, it follows that

$$y_{n_0} + \mathrm{FS}_{n_0+1}(\vec{y})$$

is not an IP-set. Now for every $y = \sum_{i\in a} y_i \in \mathrm{FS}\left((y_i)_{i\in n_0}\right)$, if $m = \min(a)$ then we have that

$$y_{n_0} + y + \mathrm{FS}_{n_0+1}(\vec{y}) \subset y_m + \mathrm{FS}_{m+1}(\vec{y})$$

and since the latter is not an IP-set (by applying the fact that ρ is a rank function to the sequence $(y_{m+k})_{k\in\omega}$), neither is the former. Finally

$$y_{n_0} + \mathrm{FS}\left((y_i)_{i\in n_0}\right)$$

is finite and hence not an IP-set. This allows one to conclude that

$$x_0 + \mathrm{FS}_1(\vec{x})$$

is not an IP-set, as claimed. $\square$

Denote in the following by $\mathbb{R}/\mathbb{Z}$ the quotient of $\mathbb{R}$ by the subgroup $\mathbb{Z}$. It is a well known fact that any commutative cancellative semigroup embeds into an abelian group, see [7, Proposition II.3.2]. Moreover any abelian group can be embedded in a divisible abelian group, and a divisible abelian group in turn can be embedded into a direct sum of copies of $\mathbb{R}/\mathbb{Z}$ (see for example [5, Theorems 24.1 and 23.1]). It follows that every cancellative commutative semigroup is a subsemigroup of a direct sum $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ of κ copies of $\mathbb{R}/\mathbb{Z}$ for some cardinal κ. Therefore it is enough to prove Proposition 3.1 for $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$. The proof of this fact will occupy the rest of this section.

Let us fix a cardinal κ. If $x \in (\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ and $\alpha \in \kappa$ then $\pi_\alpha(x)$ denotes the α-th coordinate of x. Elements of $\mathbb{R}/\mathbb{Z}$ will be freely identified with their representatives in $\mathbb{R}$ (thus we might write something like $t \neq 0$, and this really means $t \notin \mathbb{Z}$), and if we need to specify a particular representative, we will choose the unique such in $[0, 1)$. Consider the partition

$$(\mathbb{R}/\mathbb{Z})^{\oplus\kappa} = C \cup B \cup \{0\}$$

of $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$, where B is the set of elements of $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ of order 2.

Lemma 3.5. *The subset B of $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ is strongly* IP*-regular.*

Proof. Observe that $B \cup \{0\}$ is a subgroup of $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ isomorphic to the direct sum of κ copies of $\mathbb{Z}/2\mathbb{Z}$. Thus $B \cup \{0\}$ has the structure of κ-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. If $\vec{x}$ is a sequence in B then $\mathrm{FS}(\vec{x}) \cup \{0\}$ is the vector space generated by $\vec{x}$. Moreover the sequence $\vec{x}$ is linearly independent if and only if $0 \notin \mathrm{FS}(\vec{x})$ (see [4, Proposition 4.1]). Thus if $\vec{x}$ is a sequence in B such that $\mathrm{FS}(\vec{x}) \subset B$ then $\vec{x}$ is a linearly independent sequence, and hence any element of $\mathrm{FS}(\vec{x})$ can be written in a unique way as a sum of elements of the sequence $\vec{x}$. In particular $x_0 + \mathrm{FS}_1(\vec{x})$

consists of those finite sums $\sum_{i\in a} x_i$ such that $0 \in a$. Thus if a, b are finite subsets of $\omega \setminus 1$, then so is $a \triangle b$, hence

$$x_0 + \sum_{i\in a} x_i + x_0 + \sum_{i\in b} x_i = \sum_{i\in a\triangle b} x_i \notin x_0 + \mathrm{FS}_1(\vec{x}).$$

This shows that whenever $x, y \in x_0 + \mathrm{FS}_1(\vec{x})$ then $x + y \notin x_0 + \mathrm{FS}_1(\vec{x})$, which implies that $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set and B is IP-regular. □

It remains to show now that C is IP-regular. Elements $x \in C$ have order strictly greater than 2, thus there is at least one $\alpha < \kappa$ such that $\pi_\alpha(x) \notin \left\{0, \frac{1}{2}\right\}$, hence it is possible to define the function $\mu : C \to \kappa$ by

$$\mu(x) = \min\left\{\alpha \in \kappa : \pi_\alpha(x) \notin \left\{0, \frac{1}{2}\right\}\right\}.$$

Consider

$$\begin{aligned} C_1 &= \left\{x \in C : \pi_{\mu(x)}(x) = \frac{1}{4}\right\}; \\ C_3 &= \left\{x \in C : \pi_{\mu(x)}(x) = \frac{3}{4}\right\}; \\ C_2 &= \left\{x \in C : \pi_{\mu(x)}(x) \notin \left\{\frac{1}{4}, \frac{3}{4}\right\}\right\}. \end{aligned}$$

Observe that

$$C = C_1 \cup C_2 \cup C_3$$

is a partition of C.

Lemma 3.6. *The function μ restricted to C_1 is a rank function on C_1 as in Definition 3.2.*

Proof. Suppose that $\vec{x}$ is a sequence in $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ such that $\mathrm{FS}(\vec{x}) \subset C_1$. We will show that the function μ restricted to $\{x_n \big| n \in \omega\}$ is at most two-to-one, in particular finite-to-one. This is because if $n, m, k \in \omega$ are three distinct numbers such that $\mu(x_n) = \mu(x_m) = \mu(x_k) = \alpha$, then for $\beta < \alpha$ we get that $\pi_\beta(x_n+x_m+x_k)$ is an element of $\left\{0, \frac{1}{2}\right\}$ because so are $\pi_\beta(x_n), \pi_\beta(x_m), \pi_\beta(x_k)$. On the other hand, $\pi_\alpha(x_n+x_m+x_k) = \frac{3}{4}$, which shows that $\mu(x_n+x_m+x_k) = \alpha$ but $x_n+x_m+x_k \in C_3$, a contradiction.

Now assume also that $\alpha = \mu(x_0) \leq \mu(x_i)$ for every $i \in \omega$. By the previous paragraph, there is at most one $n \in \omega \setminus 1$ such that $\mu(x_n) = \mu(x_0) = \alpha$. Thus the first case is when there is such n. The first thing to notice is that for each $k \in \omega \setminus \{0, n\}$, we have that $\pi_\alpha(x_k) = 0$. This is because otherwise, since $\mu(x_k) > \alpha$ we would have that $\pi_\alpha(x_k) = \frac{1}{2}$ and so $\pi_\alpha(x_0+x_k) = \frac{3}{4}$. Therefore by an argument similar to that in the previous paragraph, we would also have $\mu(x_0+x_k) = \alpha$, which would imply that $x_0 + x_k \in C_3$, a contradiction. Now write

$$x_0+\mathrm{FS}_1(\vec{x}) = \{x_0+x_n\} \cup \big(x_0 + \mathrm{FS}((x_k)_{k\in\omega\setminus\{0,n\}})\big) \cup \big(x_0 + x_n + \mathrm{FS}((x_k)_{k\in\omega\setminus\{0,n\}})\big)$$

Clearly $\{x_0 + x_1\}$ is not an IP-set, as it is finite. Now since $\pi_\alpha(x_k) = 0$ for $i \notin \omega \setminus \{0, n\}$, it follows that every element $x \in x_0 + \mathrm{FS}((x_k)_{k\in\omega\setminus\{0,n\}})$ must satisfy $\pi_\alpha(x) = \frac{1}{4}$, which implies that $x_0+\mathrm{FS}((x_k)_{k\in\omega\setminus\{0,n\}})$ cannot contain the sum of any two of its elements and consequently it is not an IP-set. Similarly, every element

$x \in x_0 + x_n + \mathrm{FS}((x_k)_{k\in\omega\setminus\{0,n\}})$ satisfies $\pi_\alpha(x) = \frac{1}{2}$, so this set is, by the same argument, not an IP-set. Hence $x_0 + \mathrm{FS}_1(\vec{x})$ is not an IP-set.

Now if there is no such n, i.e. if $\mu(x_k) > \mu(x_0) = \alpha$ for all $k > 0$, then arguing as in the previous paragraph we get that $\pi_\alpha(x_k) = 0$ for all $k > 0$. Hence every element $x \in x_0 + \mathrm{FS}_1(\vec{x})$ satisfies that $\pi_\alpha(x) = \frac{1}{4}$, therefore the set $x_0 + \mathrm{FS}_1(\vec{x})$ cannot be an IP-set. This concludes the proof that μ is a rank function on C_1. $\square$

Considering the fact that the function $t \mapsto -t$ is an automorphism of $(\mathbb{R}/\mathbb{Z})^{\oplus\kappa}$ mapping C_1 onto C_3 and preserving μ allows one to deduce from Lemma 3.6 that μ is a rank function on C_3 as well. Thus it only remains to show that C_2 is IP-regular.

Define

$$Q_{i,j} = \left\{ x \in C_2 : \pi_{\mu(x)}(x) \in \bigcup_{m\in\omega} \left[\frac{i}{4} + \frac{1}{2^{3m+j+3}}, \frac{i}{4} + \frac{1}{2^{3m+j+2}} \right) \right\}$$

for $i \in \{0,1,2,3\}$ and $j \in \{0,1,2\}$. Observe that

$$C_2 = \bigcup_{i\in 4} \bigcup_{j\in 3} Q_{i,j}$$

is a partition of C_2. In order to conclude the proof of Proposition 3.1 it is now enough to show that for every $i \in 4$ and $j \in 3$ the set $Q_{i,j}$ is IP-regular. This will follow from Lemma 3.7 by Lemma 3.4.

Lemma 3.7. *Consider $\kappa \times \omega$ well-ordered by the lexicographic order. The function $\rho : Q_{i,j} \to \kappa \times \omega$ defined by $\rho(x) = (\mu(x), m)$ where m is the unique element of ω such that*

$$\pi_{\mu(x)}(x) \in \left[\frac{i}{4} + \frac{1}{2^{3m+j+3}}, \frac{i}{4} + \frac{1}{2^{3m+j+2}} \right)$$

is a rank function on $Q_{i,j}$.

Proof. To simplify the notation let us run the proof in the case when $i = j = 0$. The proof in the other cases is analogous. By Remark 3.3 it is enough to show that if x and y are such that $x, y, x+y \in Q_{0,0}$ then $\rho(x) \neq \rho(y)$ and $\rho(x+y) = \min\{\rho(x), \rho(y)\}$, so suppose that x,y are elements of $Q_{0,0}$ such that $x + y \in Q_{0,0}$ and assume by contradiction that $\rho(x) = \rho(y) = (\alpha, m)$. Thus

$$\pi_\alpha(x), \pi_\alpha(y) \in \left[\frac{1}{2^{3m+3}}, \frac{1}{2^{3m+2}} \right)$$

and hence

$$\pi_\alpha(x+y) \in \left[\frac{1}{2^{3m+2}}, \frac{1}{2^{3m+1}} \right).$$

If $m = 0$ then

$$\pi_\alpha(x+y) \in \left[\frac{1}{4}, \frac{1}{2} \right)$$

thus

$$x + y \in C_1 \cup Q_{1,0} \cup Q_{1,1} \cup Q_{1,3}.$$

If $m > 0$ then

$$\pi_\alpha(x+y) \in \left[\frac{1}{2^{3(m-1)+2+3}}, \frac{1}{2^{3(m-1)+2+2}} \right)$$

and therefore

$$x + y \in Q_{0,2}.$$

In either case one obtains a contradiction from the assumption that $x + y \in Q_{0,0}$. This concludes the proof that $\rho(x) \neq \rho(y)$. We now claim that $\rho(x+y) = \min\{\rho(x), \rho(y)\}$. Define $\rho(x) = (\alpha, m)$ and $\rho(y) = (\beta, n)$. Let us first consider the case when $\alpha = \beta$ and without loss of generality $m > n$. In this case

$$\pi_\xi(x+y) \in \left\{0, \frac{1}{2}\right\}$$

for $\xi < \alpha$, while

$$\pi_\alpha(x+y) \in \left[\frac{1}{2^{3m+3}} + \frac{1}{2^{3n+3}}, \frac{1}{2^{3m+2}} + \frac{1}{2^{3n+2}}\right)$$

where

$$\frac{1}{2^{3m+2}} + \frac{1}{2^{3n+2}} < \frac{1}{2^{3n+1}} < \frac{1}{2^{3(n-1)+3}}.$$

This shows that $\rho(x+y) = (\alpha, n) = \min\{\rho(x), \rho(y)\}$. Let us now consider the case when $\alpha \neq \beta$ and without loss of generality $\alpha > \beta$. In this case

$$\pi_\xi(x+y) \in \left\{0, \frac{1}{2}\right\}$$

for $\xi < \beta$ while

$$\pi_\beta(x) = 0$$

(because if not then $\pi_\beta(x) = \frac{1}{2}$ and that would imply that $x + y \in \bigcup_{j\in 3} Q_{1,j}$), and hence

$$\pi_\beta(x+y) = \pi_\beta(y).$$

This shows that $\rho(x+y) = (\beta, n) = \min\{\rho(x), \rho(y)\}$. This concludes the proof of the fact that ρ satisfies the hypothesis of Remark 3.3 and, hence, it is a rank function on $Q_{0,0}$. □

4. The class of IP-regular semigroups

In this section all semigroups will be denoted multiplicatively. Let us define $\mathcal{R}$ to be the class of all IP-regular semigroups. Observe that by Proposition 3.1 $\mathcal{R}$ contains all commutative cancellative semigroups. We will now show that $\mathcal{R}$ contains all Archimedean commutative semigroups. Recall that a commutative semigroup S is *Archimedean* if for every $a, b \in S$ there is a natural number n and an element t of S such that $a^n = bt$, see [7, Section III.1]. By [7, Proposition III.1.3] an Archimedean commutative semigroup contains at most one idempotent.

Proposition 4.1. *Archimedean commutative semigroups are* IP*-regular.*

Proof. Suppose that S is a commutative Archimedean semigroup. Let us first assume that S has no idempotent elements: In this case by [6, Proposition IV.4.1] there is a congruence $\mathcal{C}$ on S such that the quotient $S/\mathcal{C}$ is a commutative cancellative semigroup with no idempotent elements. It follows from Proposition 3.1 that $S/\mathcal{C}$ is IP-regular, and therefore S is IP-regular by Lemma 2.5. Let us consider now the case when S has a (necessarily unique) idempotent element e. Denote by H_e the maximal subgroup of S containing e. By [6, Proposition IV.2.3] H_e is an ideal of S and the quotient S/H_e is a commutative nilsemigroup, i.e. a commutative semigroup with a zero element such that every element is nilpotent. By Lemma 2.5 and Proposition 3.1 it is therefore enough to show that a commutative nilsemigroup T is IP-regular. Denote by 0 the zero element of T. We claim that

$T \setminus \{0\}$ is strongly IP-regular (as in Definition 2.1). In fact if $\vec{x}$ is a sequence in T such that $\mathrm{FP}(\vec{x})$ does not contain 0, then $x_0 \, \mathrm{FP}(\vec{x})$ is not an IP-set since x_0 is nilpotent. This concludes the proof that T is IP-regular, and hence also the proof of Proposition 4.1. □

Let us now comment on the closure properties of the class $\mathcal{R}$. By Remark 2.3, $\mathcal{R}$ is closed with respect to taking subsemigroups, and contains all finite semigroups. Moreover by Proposition 2.6 if a group G has a subnormal series with factor groups in $\mathcal{R}$, then G belongs to $\mathcal{R}$. In particular $\mathcal{R}$ contains all virtually solvable groups and their subgroups. Proposition 4.2 shows that free products of elements of $\mathcal{R}$ with no idempotent elements are still in $\mathcal{R}$.

Proposition 4.2. *Suppose that S, T are semigroups. If both S and T are* IP*-regular, and T has no idempotent elements, then the free product $S * T$ is* IP*-regular.*

Proof. Denote by T_1 the semigroup obtained from T adding an identity element 1. Consider the semigroup homomorphism from $S * T$ to T_1 sending a word w to 1 if w does not contain any letters from T, and otherwise sending w to the element of T obtained from w by erasing the letters from S and then taking the product in T of the remaining letters of w. Observe that $f^{-1}[\{1\}]$ is isomorphic to S and therefore IP-regular. The conclusion now follows from Lemma 2.5. □

The particular case of Proposition 4.2 when $S = T = \mathbb{N}$ lets us obtain that the free semigroup on 2 generators is IP-regular. Considering the function assigning to a word its length, which is a semigroup homomorphism onto $\mathbb{N}$, one can see that a free semigroup in any number of generators is IP-regular, since so is $\mathbb{N}$. Via Lemma 2.4 this observation gives a short proof of [9, Lemma 2.3].

5. The main theorem

We will now present the proof of Theorem 1.4. Suppose that S is a commutative semigroup with well-founded universal semilattice. Denote by $\mathcal{N}$ the smallest semilattice congruence on S as in [6, Proposition III.2.1]. Recall that the universal semilattice of S is the quotient $S/\mathcal{N}$ by [6, Proposition III.2.2]. Moreover by [7, Theorem III.1.2] every $\mathcal{N}$-equivalence class is an Archimedean subsemigroup of S known as an *Archimedean component* of S. Pick a nonprincipal strongly summable ultrafilter p on S. If p contains some Archimedean component of S, then p is regular by Lemma 2.4 and Proposition 4.1. Let us then assume that p does not contain any Archimedean component. Denote by $f : S \to S/\mathcal{N}$ the canonical quotient map, and by q the ultrafilter on $S/\mathcal{N}$ defined by $B \in q$ if and only if $f^{-1}[B] \in p$. By Remark 1.3, q is a nonprincipal strongly productive ultrafilter on $S/\mathcal{N}$, and moreover in order to conclude that p is regular it is enough to show that q is regular. This will follow from Lemma 5.1.

Lemma 5.1. *If Λ is a well-founded semilattice, then any nonprincipal strongly summable ultrafilter q on Λ is regular.*

Proof. We can assume without loss of generality that Λ has a maximum element $x_{\max}$. For $x \in \Lambda$, denote by $\mathrm{pred}(x)$ the set

$$\{y \in \Lambda : y \leq x \text{ and } y \neq x\}$$

of *strict* predecessors of x. We will show by well-founded induction that, for every $x \in \Lambda$, if $\mathrm{pred}(x) \in q$ then q is regular. The conclusion will follow from the observation that $\mathrm{pred}\,(x_{\max}) \in q$. Suppose that x is an element of Λ such that, for every $y \in \mathrm{pred}(x)$, if $\mathrm{pred}(y) \in q$ then q is regular. Suppose that $\mathrm{pred}(x) \in q$. If $\mathrm{pred}(x)$ witnesses the fact that q is regular, then this concludes the proof. Otherwise there is a sequence $\vec{y}$ in Λ such that $\mathrm{FP}(\vec{y}) \subset \mathrm{pred}(x)$ and $y_0\,\mathrm{FP}_1(\vec{y}) \in p$. Observing that $y_0\,\mathrm{FP}_1(\vec{y}) \subset \mathrm{pred}(y_0) \cup \{y_0\}$ allows one to conclude that $\mathrm{pred}(y_0) \in q$, where $y_0 \in \mathrm{pred}(x)$. Thus by inductive hypothesis q is regular. □

This concludes the proof of the fact that a nonprincipal strongly summable ultrafilter on a commutative semigroup with well-founded universal semilattice is regular. We will now show that the same fact holds for solvable inverse semigroups with well-founded semilattice of idempotents. An introduction to inverse semigroups can be found in [6, Chapter VII] or in the monograph [14]. Recall that the semilattice of idempotents of a commutative inverse semigroup is isomorphic to its universal semilattice. The notion of solvable inverse semigroup has been introduced by Piochi in [15] as a generalization of the notion of solvable group to the context of inverse semigroups (solvable groups are thus exactly the solvable inverse semigroups with only one idempotent, see [15, Theorem 3.4]). Observe that by definition a solvable inverse semigroup S of class $n+1$ has a commutative congruence γ_S such that, if $f : S \to S/\gamma_S$ is the canonical quotient map, then

$$f^{-1}\left[E\left(S/\gamma_S\right)\right]$$

is an inverse subsemigroup of S of solvability class n. Moreover the solvable inverse semigroups of solvability class 1 are exactly the commutative semigroups. The fact that solvable inverse semigroups with well-founded semilattice of idempotents satisfy the conclusion of Theorem 1.4 will then follow from Remark 1.3 by induction on the solvability class, after we observe that a homorphic image of a semigroup with well-founded semilattice of idempotents also has a well-founded semilattice of idempotents. This is the content of Lemma 5.2.

Lemma 5.2. *Suppose that S, T are semigroups, and $f : S \to T$ is a surjective semigroup homomorphism. If S is an inverse semigroup with well-founded semilattice of idempotents, then so is T.*

Proof. By [3, Theorem 7.32] T is an inverse semigroup. If B is a nonempty subset of the idempotent semilattice $E\,(T)$ of T, let A be the set of idempotent elements a of S such that $f(a) \in B$. Since by hypothesis the idempotent semilattice $E(S)$ of S is well-founded, A has a minimal element a_0. We claim that $b_0 = f(a_0)$ is a minimal element of B. Suppose that $b \in B$ is such that $b \leq b_0$. By [14, Chapter 1, Proposition 21(3)] there exists $a \in A$ such that $f(a) = b \leq b_0 = f(a_0)$. Hence by [14, Chapter 1, Proposition 21(7)] there exists $a' \in A$ such that $a' \leq a_0$ and $f(a') = f(a) = b$. Since a is a minimal element of A we have that $a' = a_0$ and hence $b = f(a') = f(a_0) = b_0$. This concludes the proof that B has a minimal element, and that $E\,(T)$ is well-founded. □

6. Sparseness

A strongly productive ultrafilter p on a (multiplicatively denoted) semigroup S is *sparse* (see [9, Definition 3.9]) if for every $A \in p$ there are a sequence $\vec{x} = (x_n)_{n\in\omega}$ in S and a subsequence $\vec{y} = (x_{k_n})_{n\in\omega}$ of $\vec{x}$ such that:

- $\mathrm{FP}(\vec{y}) \in p$;
- $\mathrm{FP}(\vec{x}) \subset A$;

- $\{k_n : n \in \omega\}$ is coinfinite in ω.

Suppose that $\mathbb{F}$ is the partial semigroup of finite nonempty subsets of ω, where, for $a, b \in \mathbb{F}$ the product ab is defined and equal to $a \cup b$ if and only if $\max(a) < \min(b)$. A strongly productive ultrafilter on the partial semigroup $\mathbb{F}$ is an *ordered union ultrafilter* as defined in [1, page 92]. A strongly productive ultrafilter p on a multiplicatively denoted semigroup S is *multiplicatively isomorphic to an ordered union ultrafilter* if there is a sequence $\vec{x}$ such that the function

$$\begin{aligned} f : \quad \mathbb{F} \to & \quad \mathrm{FP}(\vec{x}) \\ a \mapsto & \quad \prod_{i \in a} x_i \end{aligned}$$

is injective, and furthermore

$$\left\{ f^{-1}[A] : A \in p \right\}$$

is an ordered union ultrafilter.

Lemma 6.1. *If p is multiplicatively isomorphic to an ordered union ultrafilter, then p is sparse strongly productive. In particular every ordered union ultrafilter is sparse.*

Proof. Suppose that the sequence $\vec{x}$ in S and the function $f : \mathbb{F} \to \mathrm{FP}(\vec{x})$ witness the fact that p is multiplicatively isomorphic to an ordered union ultrafilter. Fix an element B of p, and observe that

$$q = \left\{ f^{-1}\left[A \cap B\right] : A \in p \right\}$$

is an ordered union ultrafilter. Therefore there is a sequence $\vec{b}$ in $\mathbb{F}$ such that all the products from $\vec{b}$ are defined (equivalently, $\max(b_i) < \min(b_{i+1})$ for every $i \in \omega$), and $\mathrm{FP}(\vec{b}) \in q$. Moreover by [13, Theorem 4] (see also [11, Theorem 2.6]) there is an element W of q contained in $\mathrm{FP}(\vec{b})$ such that $\bigcup W$ has infinite complement in $\bigcup_{i \in \omega} b_i$. Denote by D the set of $i \in \omega$ such that $b_i \subset \bigcup W$. Observe that

$$\bigcup_{i \in D} b_i = \bigcup W$$

and

$$W \subset \mathrm{FP}\left((b_i)_{i \in D}\right).$$

In particular D has infinite complement in ω and $\mathrm{FP}\left((b_i)_{i \in D}\right)$ belongs to q. Therefore the sequence $\overrightarrow{x}$ in S such that $x_i = f(b_i)$ for every $i \in \omega$ is such that $\mathrm{FP}(\vec{x}) \subset B$ and $\mathrm{FP}\left((x_i)_{i \in D}\right) \in p$, witnessing the fact that p is sparse strongly productive. $\square$

We will now define a condition on sequences that ensures the existence of a multiplicative isomorphism with an ordered union ultrafilter. This can be seen as a noncommutative analogue of the notion of *strong uniqueness of finite sums* introduced in [11, Definition 3.1] in a commutative context.

Definition 6.2. A sequence $\vec{x}$ in a semigroup S satisfies the *ordered uniqueness of finite products* if the function

$$\begin{aligned} f : \mathbb{F} & \quad \to \quad \mathrm{FP}(\vec{x}) \\ a & \quad \mapsto \quad \prod_{i \in a} x_i \end{aligned}$$

is an isomorphism of partial semigroups from $\mathbb{F}$ to $\mathrm{FP}(\vec{x})$. Equivalently f is injective and if a,b are elements of $\mathbb{F}$ such that $f(a)f(b) \in \mathrm{FP}(\vec{x})$, then the maximum element of a is strictly smaller than the minimum element of b.

For example suppose that S is the free semigroup on countably many generators $\{s_n : n \in \omega\}$. It is not difficult to see that the sequence $(s_n)_{n\in\omega}$ in S satisfies the ordered uniqueness of finite products..

Remark 6.3. If a strongly productive ultrafilter p on S contains $\mathrm{FP}(\vec{x})$ for some sequence $\vec{x}$ in S satisfying the ordered uniqueness of finite products, then p is multiplicatively isomorphic to an ordered union ultrafilter.

Remark 6.3 follows immediately from the fact that an ordered union ultrafilter is just a strongly productive ultrafilter on the partial semigroup $\mathbb{F}$.

The following immediate consequence of Remark 6.3 and Lemma 6.1 can be seen as a noncommutative analogue of [11, Theorem 3.2] (see also [4, Corollary 2.9]).

Corollary 6.4. *Let p be a strongly productive ultrafilter on a semigroup S. If p contains $\mathrm{FP}(\vec{x})$ for some sequence $\vec{x}$ satisfying the ordered uniqueness of finite products, then p is sparse.*

In the remainder of this section, we will present an application of Corollary 6.4 to a question of Neil Hindman and Lakeshia Legette Jones from [9] about *very strongly productive* ultrafilters on the free semigroup on countably many generators.

Recall that a sequence $\vec{y}$ on a semigroup S is a *product subsystem* of the sequence $\vec{x}$ in S if there is a sequence $(a_n)_{n\in\omega}$ in $\mathbb{F}$ such that $y_n = \prod_{i\in a_n} x_i$ and the maximum element of a_n is strictly smaller than the minimum element of a_{n+1} for every $n \in \omega$. Suppose that S is the free semigroup on countably many generators, and $\vec{s}$ is an enumeration of its generators. A *very strongly productive ultrafilter* on S as in [9, Definition 1.2] is an ultrafilter p on S generated by sets of the form $\mathrm{FP}(\vec{x})$ where $\vec{x}$ is a product subsystem of $\vec{s}$.

Theorem 6.5. *Every very strongly productive ultrafilter on the free semigroup S is multiplicatively isomorphic to an ordered union ultrafilter, and hence sparse.*

Proof. Observe that by [9, Theorem 4.2] very strongly productive ultrafilters on S are exactly the strongly productive ultrafilters containing $\mathrm{FP}(\vec{s})$ as an element. In particular, since the sequence $\vec{s}$ satisfies the ordered uniqueness of finite products, all very strongly productive ultrafilters on S are multiplicatively isomorphic to ordered union ultrafilters by Remark 6.3, and hence sparse by Lemma 6.1. □

Theorem 6.5 answers Question 3.26 from [9]. Corollary 3.11 of [9] asserts that a sparse very strongly productive ultrafilter on S can be written only trivially as a product of ultrafilters on the free group on the same generators. Since by Theorem 6.5 any very strongly productive ultrafilter on S is sparse, one can conclude that the conclusion of [9, Corollary 3.11] holds for any very strongly productive ultrafilter on S. This is the content of Corollary 6.6.

Corollary 6.6. *Let G be the free group on the sequence of generators $\vec{s}$, and let S be the free semigroup on the same generators. Suppose that p is a very strongly productive ultrafilter on S. If q,r are ultrafilters on G such that $qr = p$, then there is an element w of G such that one of the following statements hold:*

(1) *$r = wp$ and $q = pw^{-1}$;*

(2) *$r = w$ and $q = pw^{-1}$;*
(3) *$r = wp$ and $q = w^{-1}$.*

In particular, if $q, r \in G^*$ are such that $qr = p$, then $r = wp$ and $q = pw^{-1}$ for some $w \in G$.

References

[1] Andreas Blass. Ultrafilters related to Hindman's finite-unions theorem and its extensions. In *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 263–274. Amer. Math. Soc., Providence, RI, 1987.

[2] V. Capraro and M. Lupini, with an appendix by Vladimir Pestov. Introduction to sofic and hyperlinear groups and Connes' Embedding Conjecture . *Lecture Notes in Mathematics*, Springer (in press).

[3] A. H. Clifford and G. B. Preston. The algebraic theory of semigroups. Vol. II. *Mathematical Surveys*, No. 7 American Mathematical Society, Providence, R.I. 1967

[4] David Fernández Bretón. Strongly summable ultrafilters, union ultrafilters, and the trivial sums property. To appear in *Canad. J. Math.* (available online at http://dx.doi.org/10.4153/CJM-2015-023-9).

[5] László Fuchs. Infinite abelian groups. Vol. I. *Pure and Applied Mathematics*, Vol. 36. Academic Press, New York-London, 1970.

[6] Pierre Antoine Grillet. Semigroups. An introduction to the structure theory, volume 193 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1995.

[7] Pierre Antoine Grillet. Commutative semigroups, volume 2 of *Advances in Mathematics (Dordrecht)*. Kluwer Academic Publishers, Dordrecht, 2001.

[8] Neil Hindman. Summable ultrafilters and finite sums. In *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 263–274. Amer. Math. Soc., Providence, RI, 1987.

[9] Neil Hindman and Lakeshia Legette Jones. Idempotents in βS that are only products trivially. *New York J. Math.* 20 (2014), 57–80

[10] Neil Hindman, Igor Protasov, and Dona Strauss. Strongly summable ultrafilters on abelian groups. *Mat. Stud.*, 10(2):121–132, 1998.

[11] Neil Hindman, Juris Steprāns, and Dona Strauss. Semigroups in which all strongly summable ultrafilters are sparse. *New York J. Math.*, 18:835–848, 2012.

[12] Neil Hindman and Dona Strauss. Algebra in the Stone-Čech compactification: theory and applications. 2nd revised and extended ed. *De Gruyter Textbook*. Walter de Gruyter & Co., Berlin, 2012.

[13] Peter Krautzberger. On strongly summable ultrafilters. *New York J. Math.*, 16:629–649, 2010.

[14] Mark V. Lawson. Inverse semigroups. The theory of partial symmetries. *World Scientific Publishing Co. Inc.*, River Edge, N.J., 1998.

[15] Brunetto Piochi. Solvability in inverse semigroups. *Semigroup Forum*, 34:287–303, 1986.

[16] William R. Scott. Group theory. *Prentice-Hall Inc.*, Englewood Cliffs, N.J., 1964.

Department of Mathematics and Statistics, York University, N520 Ross, 4700 Keele Street, M3J 1P3, Toronto (Ontario), Canada

Current address: Mathematics Department, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, U.S.A.

E-mail address: `djfernan@umich.edu`

URL: `http://www-personal.umich.edu/~djfernan/`

Department of Mathematics and Statistics, York University, N520 Ross, 4700 Keele Street, M3J 1P3, Toronto (Ontario), Canada

Current address: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, Room 02.126, 1090 Wien, Austria.

E-mail address: `martino.lupini@univie.ac.at`

URL: `www.lupini.org`