DISSERTATION SUBJECT ORAL SYLLABUS

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1. **INFINITARY COMBINATORICS**[6, 13, 14, 16, 18]
   (a) Definition of a Δ-system, statement and proof of the Δ-system lemma.
   (b) Statement of MA (Martin's Axiom) and MA_κ, proof that MA_ω holds and $\mathsf{MA}_\mathfrak{c}$ fails.
   (c) Definition of a club and a stationary subset of κ, of the club filter and proof that the club filter on a regular cardinal κ is κ-complete.
   (d) Statement and proof of the Pressing Down Lemma.
      i. An application: Statement and proof of Thomas's theorem on automorphism towers of centreless groups.
   (e) Statement of $\diamondsuit$ (the Diamond Axiom) and $\clubsuit$ (the Club Axiom).
   (f) Trees
      i. Statement and proof of König's Lemma.
      ii. Definition of a κ-tree, a κ-Aronszajn tree and a κ^+-Kurepa tree.
      iii. Proof of existence of an ω_1-Aronszajn tree.
      iv. Definition of a κ-Suslin tree, proof that κ-Suslin implies κ-Aronszajn.
      v. Definition of a Suslin line and of Suslin's hypothesis SH, proof that the existence of a Suslin tree is equivalent to the existence of a Suslin Line.
      vi. MA_{ω_1} implies that there is no ω_1-Suslin tree.
      vii. $\diamondsuit$ implies existence of a Suslin Tree.
   (g) Definition of $\mathrm{Def}(X)$ (Definable subsets of a set X), the constructible hierarchy, definition of $\mathbf{L}$, the Axiom of Constructibility $\mathbf{V} = \mathbf{L}$.
   (h) Statement only: $\mathbf{L} \models \mathbf{V} = \mathbf{L}$, $\mathbf{V} = \mathbf{L}$ implies that there is a definable well-order of the universe; the Condensation Lemma.
   (i) $\mathbf{L} \models \mathsf{AC} + \mathsf{GCH}$, $\mathbf{L} \models \diamondsuit$.

2. **LARGE CARDINALS**[6, 13, 15]
   (a) Definition of a weakly inaccessible and a strongly inaccessible cardinal.
   (b) If κ is a strongly inaccessible cardinal, then $V_\kappa = H(\kappa) \models \mathsf{ZFC}$.
   (c) The existence of a weakly inaccessible cardinal is equiconsistent with that of a strongly inaccessible one, which in turn has consistency strength strictly greater than that of ZFC.
   (d) Definition of measurable cardinals, proof that measurable cardinals are strongly inaccessible.
   (e) Proof that κ is a measurable cardinal iff there exists a nontrivial elementary embedding $j : \mathbf{V} \longrightarrow M$, with critical point κ, into some transitive class M.
   (f) If there exists a measurable cardinal then $\mathbf{V} \neq \mathbf{L}$.
   (g) Definition of a supercompact cardinal and characterization in terms of elementary embeddings.
   (h) Definition of a Reinhardt cardinal, proof of the nonexistence of such cardinals (Kunen's inconsistency).
   (i) Statement of the axioms I1, I2, I3.

3. **FORCING**[1, 2, 3, 7, 8, 12, 13, 16, 17, 20]

   (a) Definitions of $\mathbb{P}$-generic filter, the forcing language, the forcing relation, antichains, nice name, dense below p, predense, the κ-c.c., the Knaster condition, separative and non-atomic partial orders, λ-closed, complete embedding, dense embedding.

   (b) Proof that κ-c.c. forcing notions preserve cofinalities and cardinals $\geq \kappa$ and κ-closed ones preserve cofinalities and cardinals $\leq \kappa$.

   (c) Definition of strategically σ-closed forcing notion, proof that such notions don't add reals.

   (d) Cohen forcing

      i. Proof of the consistency of $\neg\mathsf{CH}$.
      ii. How to use forcing to obtain a model of $\neg\mathsf{AC}$.
      iii. Every countable forcing notion is equivalent to adding a Cohen real.
      iv. In Cohen's model, $\omega_1 = \mathfrak{a}$ and $\mathfrak{d} = \mathfrak{c}$.
      v. Proof that in Cohen's model, Borel's conjecture does not hold.
      vi. Statement of the dual Borel's conjecture, proof that in Cohen's model the dual Borel conjecture holds.

   (e) How to force CH, and $\diamondsuit$.

   (f) The Lévy collapse.

   (g) Forcing with a Suslin tree destroys its Suslinness.

   (h) Namba forcing.

   (i) Tree Prikry forcing.

   (j) Iterated Forcing.

      i. Definitions: α-stage iteration, direct limit, inverse limit, finite support iteration, countable support iteration.
      ii. The finite support iteration of Cohen forcings is forcing equivalent to the finite support product of Cohen forcings.
      iii. Finite support iterations add Cohen reals at limit stages.
      iv. Proof of the consistency of $\mathsf{MA} + \neg\mathsf{CH}$.

   (k) Proper Forcing.

      i. Definitions of $(\mathbb{P}, M)$-generic, and of a proper forcing notion.
      ii. Countably closed forcing notions and c.c.c. forcing notions are proper.
      iii. Proper forcing notions preserve stationary subsets of $[\omega_1]^\omega$.
      iv. Characterization of proper forcing notions in terms of games.
      v. Sacks forcing, Mathias forcing, Laver forcing and Grigorieff forcing are proper.
      vi. Proof of the preservation theorem: countable support iterations of proper forcings are proper.
      vii. An application: a model where Borel's conjecture holds.
      viii. Another application: a model with no P-points.

   (l) The Proper Forcing Axiom PFA.

      i. Statement of PFA.
      ii. PFA is consistent provided that a supercompact cardinal exists.

4. **CARDINAL INVARIANTS**[4, 11, 19]

   (a) Definitions of an unbounded, dominating, centred, AD and MAD family; definition of a scale, a pseudointersection, a tower.

   (b) Definitions of $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{p}, \mathfrak{t}, \mathfrak{m}$ and variations of $\mathfrak{m}$.

   (c) Proof that $\omega_1 \leq \mathfrak{m} \leq \mathfrak{m}(\sigma{-}\text{centred}) \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$, proof that $\mathfrak{b} \leq \mathfrak{a}$.

   (d) Proof that there exists a scale iff $\mathfrak{b} = \mathfrak{d}$.

   (e) Proof that $\mathfrak{t}$ and $\mathfrak{b}$ are regular, and $\text{cf}(\mathfrak{d}) \geq \mathfrak{b}$.

(f) $\text{cov}(\mathscr{M}) = \mathfrak{m}(\text{countable}) = \mathfrak{m}(\text{Cohen})$.

(g) $\mathfrak{p} = \mathfrak{m}(\sigma{-}\text{centred})$, and $\mathfrak{p}$ is regular.

5. **ULTRAFILTERS ON SEMIGROUPS**[2, 5, 9, 10]

   (a) Definition of filter, ultrafilter.

   (b) Definition of a P-point, a Q-point, a selective (or Ramsey) ultrafilter and some characterizations of these.

   (c) The Stone-Čech compactification of ω.

   i. The topology of $\beta\omega$.

   ii. $\beta\omega$ is compact Hausdorff, totally disconnected, of cardinality 2^{2^ω} and has no nontrivial convergent sequences.

   iii. Characterization of open and closed subsets of $\beta\omega$ as ideals and filters, respectively.

   iv. The extension property, extension of a semigroup operation.

   (d) The semigroup $\beta\omega$.

   i. The Ellis-Nukamura lemma.

   ii. Hindman's finite sums theorem.

   iii. Definition of strongly summable and of weakly summable ultrafilters.

   iv. On abelian groups, strongly summable implies idempotent, which in turn implies weakly summable.

   v. Definition of a union ultrafilter.

   vi. Definition of an additive isomorphism between ultrafilters.

   vii. Every strongly summable ultrafilter on $(\omega, +)$ is additively isomorphic to a union ultrafilter, and viceversa.

   viii. Definition of sparse strongly summable ultrafilters.

   ix. MA implies the existence of sparse strongly summable ultrafilters over any abelian group.

   x. P-points are points of continuity of the function $q + (\ )$ for any $q \in \beta\omega$.

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