A PARAMETRIZED DIAMOND PRINCIPLE
AND UNION ULTRAFILTERS

BY

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Abstract. We consider a cardinal invariant closely related to Hindman's theorem. We prove that this invariant is small in the iterated Sacks perfect set forcing model, and its corresponding parametrized diamond principle implies the existence of union ultrafilters. As a corollary, we obtain the existence of union ultrafilters in the iterated Sacks model of set theory.

1. Introduction. Union ultrafilters are ultrafilters that relate naturally to Hindman's theorem (in much the same way as Ramsey ultrafilters relate naturally to Ramsey's theorem). Recall that Hindman's theorem [14, Theorem 3.1], in one of its equivalent forms (the equivalence, which is explicitly pointed out in [2, p. 384], can be seen via the identification of each natural number with the finite set of places where its binary expansion has a nonzero digit), states that whenever we partition the set $[\omega]^{<\omega}$ of all finite subsets of ω into two cells, $[\omega]^{<\omega} = A_0 \cup A_1$, there exists an infinite pairwise disjoint family $X \subseteq [\omega]^{<\omega}$ such that the set

$$\mathrm{FU}(X) = \left\{ \bigcup_{x \in F} x \ \Big| \ F \in [X]^{<\omega} \setminus \{\emptyset\} \right\}$$

is contained in a single A_i. Union ultrafilters were first introduced by Blass [4, p. 92] (^{1}), and have been studied by several people afterwards, including the first author of this paper [5, 6, 9, 10, 11, 15, 17].

(^{1}) A close relative of union ultrafilters, known as *strongly summable ultrafilters*, were introduced at around the same time by Hindman [13, Definition 2.1], who had actually proved earlier that their existence follows from the continuum hypothesis CH [12, Theorem 3.3]. On the other hand, Matet [20] arrived independently at an equivalent concept, in terms of what he called *filters of partitions*. Specifically, he established several facts about objects he calls *Hindman filters* (union ultrafilters) and *Milliken–Taylor filters* (stable ordered union ultrafilters).

DEFINITION 1.1. An ultrafilter u on $[\omega]^{<\omega}$ is said to be a *union ultrafilter* if, for every $A \in u$, there exists an infinite, pairwise disjoint family $X \subseteq [\omega]^{<\omega}$ such that $A \subseteq \mathrm{FU}(X) \ni u$.

One of the most interesting questions regarding union ultrafilters is their existence. The existence of union ultrafilters [2] follows from the Continuum Hypothesis, abbreviated CH (i.e. the statement that the set of real numbers is in bijection with the first uncountable cardinal, or $|\mathbb{R}| = \omega_1$) by [12, Theorem 3.3]. On the other hand, Blass and Hindman [6, Theorem 2] proved that the existence of a union ultrafilter implies the existence of a P-point, and hence cannot be established within ZFC, by Shelah's result on the independence of the existence of P-points from ZFC [3] [26, Theorem 6.5]. These existential results turn out to be relevant to the study of extremally disconnected topological groups, since union ultrafilters naturally give rise to such groups. Specifically, if u is a union ultrafilter, then declaring $\{A \cup \{\emptyset\} \mid A \in u\}$ to be the neighborhood filter of the identity element $\emptyset$ yields an extremally disconnected (even maximal) group topology on the Boolean group $[\omega]^{<\omega}$, and hence provides consistent examples of such groups [1, 19, 23, 25, 24].

From the set-theoretic viewpoint, it is relevant to determine in which of the many models of set theory that can be obtained by forcing, union ultrafilters exist. For example, Eisworth [9, Theorem 9] established that the existence of union ultrafilters follows from the equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ (equivalent to Martin's Axiom restricted to countable forcing notions), and consequently all models satisfying this equality (this includes Cohen's classical model, as well as all models obtained by finite support iterations of ccc forcing notions whose length equals $\mathfrak{c}$ in the final extension) will satisfy such an existential statement. The first author [10, Section 4.3] has constructed models of ZFC that contain union ultrafilters, even though $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$ in those models. On the other hand, there are also results establishing that the existence of union ultrafilters implies the existence of other set-theoretic objects such as semiselective ultrafilters [20, Proposition 7.5], or pairs of non-near-coherent ultrafilters [5, Theorem 38], which allow one to conclude that many of the better-studied *canonical models* [4] do not have union ultrafilters: Miller's model because it satisfies the Near-Coherence of Filters principle [7, Theorem, p. 530], Laver's and Mathias's models because in those models, rapid

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[2] And consequently also the existence of strongly summable ultrafilters, by [6, Theorem 1].

[3] Quite recently, a much simpler proof of the consistency of no P-points has been found by Chodounský and Guzmán [8], who proved that there are no P-points in the iterated Silver model and also in the side-by-side Silver model.

[4] Here, a *canonical model* is a model of ZFC obtained by means of a countable support iteration of proper Borel forcing notions.

ultrafilters do not exist [21, Theorem 5; Remark (1) on p. 106]; or the iterated Silver model, because by the recent result of [8], this model has no P-points. Another model (though not canonical) that has been studied extensively is the Random model, where union ultrafilters cannot exist either, because this model does not have semiselective ultrafilters [18, Theorem 5.1] (that is, rapid P-points).

The very interesting question of whether the iterated Sacks model (obtained by a countable support iteration of perfect set forcings) satisfies such an existential statement is not immediately answered with these sorts of considerations, and was actually explicitly asked by the first author [10, Question 4.11, p. 106]. We are able to answer this question in the affirmative by using the theory of parametrized diamond principles, applied to a particular cardinal characteristic of the continuum, and this is precisely the main result of this paper. It should be noted that a result of Zheng [29, Theorem 0.3], obtained independently and at around the same time, provides an alternative proof for the main result of this paper ([5]).

In Section 2 we define the cardinal invariant that we will use, along with its parametrized diamond principle, and show that the invariant is small in the iterated Sacks model, which in turn implies that the corresponding diamond principle holds in the iterated Sacks model. Then in Section 3 we prove that the aforementioned parametrized diamond principle implies the existence of a union ultrafilter.

2. A cardinal invariant, its parametrized diamond principle, and the iterated Sacks model. In this section we will start by considering the following cardinal characteristic of the continuum, whose definition arises directly from Hindman's theorem. Our notation for this invariant follows [3].

DEFINITION 2.1. The cardinal invariant $\mathfrak{hom}_H$ is defined to be the least cardinality of a family $\mathcal{X}$ of infinite ordered ([6]) subfamilies of $[\omega]^{<\omega}$ such that for every colouring of $[\omega]^{<\omega}$ into finitely many colours, there exists an $X \in \mathcal{X}$ such that $\mathrm{FU}(X)$ is monochromatic.

Since within every set of the form $\mathrm{FU}(X)$ we can find infinite ordered subfamilies, in the above definition we could drop the requirement that every element of $\mathcal{X}$ be ordered, and we would still obtain an equivalent cardinal. In

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([5]) Strictly speaking, Zheng proves that there are union ultrafilters in the *side-by-side* Sacks model, although it seems plausible that her argument might be suitably modified to prove the existence of such ultrafilters in the iterated Sacks model too. In any case, our proof is substantially different from Zheng's, and arguably interesting in its own right.

([6]) A family $X \subseteq [\omega]^{<\omega}$ is said to be *ordered* if, for every $x, y \in X$, either $\max(x) < \min(y)$ or $\max(y) < \min(x)$. Ordered families are sometimes called *block sequences*. Throughout this paper, we will use both terminologies indistinctly.

fact, by using symmetric differences instead of unions (for a pairwise disjoint family, the two approaches are exactly the same), we could even drop the requirement that every element of $\mathcal{X}$ consists of pairwise disjoint sets, and we would still get the exact same cardinal invariant.

In [22], a general theory of weak diamond principles is studied. The cardinal invariants considered there are defined in terms of triples (A, B, E) where $|A| \leq \mathfrak{c}$, $|B| \leq \mathfrak{c}$ and $E \subseteq A \times B$, satisfying $(\forall a \in A)(\exists b \in B)(a\ E\ b)$ and $(\forall b \in B)(\exists a \in A)\neg(a\ E\ b)$ (the reason for the last two requirements is just that we can thus ensure the existence and nontriviality of the corresponding cardinal invariant). The cardinal invariant associated to such a triple (A, B, E) (sometimes referred to as the *evaluation* of the triple), which we denote by $\langle A, B, E \rangle$, is given by

$$\langle A, B, E \rangle = \min\{|X| \mid X \subseteq B \wedge (\forall a \in A)(\exists b \in X)(a\ E\ b)\}.$$

We are especially interested in cardinal invariants $\langle A, B, E \rangle$ that are Borel, i.e. all entries of the triple (A, B, E) can be viewed as Borel subsets of some Polish space. For a cardinal invariant of this type, the *parametrized diamond principle* corresponding to the triple is the statement that for every Borel function $F : 2^{<\omega_1} \to A$ (where Borel just means that for every $\alpha < \omega_1$ the restriction $F{\upharpoonright}2^{<\alpha}$ is a Borel function) there exists a $g : \omega_1 \to B$ such that for every $f : \omega_1 \to 2$, the set

$$\{\alpha < \omega_1 \mid F(f{\upharpoonright}\alpha)\ E\ g(\alpha)\}$$

is stationary. This statement is denoted by $\diamondsuit(A, B, E)$. The fundamental theorem regarding these parametrized diamond principles is the following:

THEOREM 2.2 ([22, Theorem 6.6]). *Let (A, B, E) be a Borel triple defining a cardinal invariant. Let $\langle \mathring{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ be a sequence of (names for) Borel forcing notions that satisfy $\Vdash_\alpha$ "$\mathring{\mathbb{Q}}_\alpha \cong \check{2}^+ \times \mathring{\mathbb{Q}}_\alpha$" (here 2^+ denotes the partial order that consists of two incomparable elements, plus one maximal element $\mathbb{1}$ above them), and assume that the countable support iteration $\mathbb{P}_{\omega_2}$ of these forcing notions is proper. Then*

$$\mathbb{P}_{\omega_2} \Vdash \text{``}\diamondsuit(A, B, E)\text{''} \quad \textit{if and only if} \quad \mathbb{P}_{\omega_2} \Vdash \text{``}\langle A, B, E \rangle \leq \omega_1\text{''}.$$

All of the above is relevant to our cardinal invariant $\mathfrak{hom}_H$ mainly because its definition can be thought of as given by

$$\mathfrak{hom}_H = \langle \wp([\omega]^{<\omega}), \{X \in \wp([\omega]^{<\omega}) \mid X \text{ is a block sequence}\},$$

$$\text{``is FU-reaped by''} \rangle$$

where a set A is FU-*reaped* by a block sequence X if either $\mathrm{FU}(X) \subseteq A$, or $\mathrm{FU}(X) \cap A = \emptyset$ (i.e. $\mathrm{FU}(X)$ is homogeneous for the colouring $\{A, [\omega]^{<\omega} \setminus A\}$). So in this context, the combinatorial principle $\diamondsuit(\mathfrak{hom}_H)$ is the statement that for every $F : 2^{<\omega_1} \to \wp([\omega]^{<\omega})$, there exists an ω_1-sequence g of block sequences such that, for every $f : \omega_1 \to 2$, $F(f{\upharpoonright}\alpha)$ is FU-reaped by $g(\alpha)$

(i.e. $\mathrm{FU}(g(\alpha))$ reaps $F(f{\restriction}\alpha)$) for stationarily many $\alpha < \omega_1$. And in order to see that such a principle holds in a forcing extension (such as the one obtained by iterating Sacks forcing) obtained by a countable support iteration of proper forcing notions, it will suffice to establish that in such an extension the cardinal $\mathfrak{hom}_H$ equals ω_1.

So our next immediate goal is to establish that $\mathfrak{hom}_H = \omega_1$ in the iterated Sacks model, and in order to attain this goal, we now state some definitions and results from [28], which will be used for this purpose.

DEFINITION 2.3 ([28, Def. 6.1.9]). A cardinal invariant $\mathfrak{x}$ is said to be *very tame* if its definition is of the form

$$\mathfrak{x} = \min\{|X| \mid \phi(X) \wedge (\forall x \in \mathbb{R})(\exists y \in X)(\theta(x, y))\},$$

where

- $\phi(X)$ is a formula of the form

$$(\forall x_0, \ldots, x_k \in X)(\exists y_0, \ldots, y_l \in X)(\psi(x_0, \ldots, x_k, y_0, \ldots, y_l))$$

  for some arithmetic formula ([7]) ψ;
- $\theta(x, y)$ is an analytic ([8]) formula;
- $\mathsf{ZFC} \vdash (\forall X \subseteq \mathbb{R})((|X| = \omega \wedge \phi(X)) \Rightarrow (\exists Z)(X \subseteq Z \wedge |Z| = \mathfrak{x} \wedge Z$ witnesses $\mathfrak{x}$)).

LEMMA 2.4. $\mathfrak{hom}_H$ *is very tame.*

Proof. We start by canonically identifying ω with $[\omega]^{<\omega}$ (during the proof, at some point we will need such an identification to be expressible by a Δ_0^0 formula, which can be done, for example, by means of binary expansions), so that we have a corresponding coding of $\wp([\omega]^{<\omega})$ by means of $\mathbb{R}$. Thus, we just need to find a formula $\phi(\mathscr{X})$ of the form $(\forall x_0, \ldots, x_k \in \mathscr{X})(\exists y_0, \ldots, y_l \in \mathscr{X})(\psi(x_0, \ldots, x_k, y_0, \ldots, y_l))$ for some arithmetic ψ; as well as an analytic formula θ such that

$$\mathfrak{hom}_H = \min\{|\mathscr{X}| \mid \phi(\mathscr{X}) \wedge (\forall A \in \wp([\omega]^{<\omega}))(\exists X \in \mathscr{X})(\theta(A, X))\},$$

and

$$\mathsf{ZFC} \vdash (\forall \mathscr{X} \subseteq \wp([\omega]^{<\omega}))((|\mathscr{X}| = \omega \wedge \phi(\mathscr{X})) \Rightarrow$$
$$(\exists \mathscr{Z})(\mathscr{X} \subseteq \mathscr{Z} \wedge |\mathscr{Z}| = \mathfrak{hom}_H \wedge \mathscr{Z} \text{ witnesses } \mathfrak{hom}_H)).$$

Since $\mathfrak{hom}_H$ is given by

$$\mathfrak{hom}_H = \min\{|\mathscr{X}| \mid \mathscr{X} \text{ consists of infinite block sequences } \wedge$$
$$(\forall A \in \wp([\omega]^{<\omega}))(\exists X \in \mathscr{X})(X \text{ FU-reaps } A)\},$$

([7]) An *arithmetic* formula is one all of whose quantifiers are bounded by ω, or in other words, all of its quantifiers range only over natural numbers.

([8]) An *analytic* formula is one that consists of a quantifier ranging over real numbers (or over subsets of ω, or any other equivalent object) followed by an arithmetic formula.

it suffices to show that the formula $\phi(\mathscr{X})$ expressing that $\mathscr{X}$ is a family of infinite block sequences, as well as the formula $\theta(A, X)$ expressing that A is FU-reaped by X, are as required.

We first look at the formula $\psi(X)$ which expresses that X is an infinite block sequence. Such a formula, when written down explicitly, reads

$$(\forall x \in X)(\forall y \in X)(\max(x) < \min(y) \vee \max(y) < \min(x))$$
$$\wedge (\forall x \in X)(\exists y \in X)(\max(x) < \min(y)).$$

Since the quantifiers of this formula range only over elements of $[\omega]^{<\omega}$ (which are just natural numbers according to our coding), the formula is arithmetic. Now, the formula $\phi(\mathscr{X})$ expressing that $\mathscr{X}$ is a family of infinite block sequences is simply $(\forall X \in \mathscr{X})(\psi(X))$, so it is as required.

Secondly, the formula $\theta(A, X)$ is given by

$$(\forall F \in [X]^{<\omega})(\bigcup F \in A) \vee (\forall F \in [X]^{<\omega})(\bigcup F \notin A),$$

which is clearly analytic, as it contains only one quantifier (and $\bigcup F$ can be defined by a Δ_0^0 formula).

Finally, it is trivial that every countable family $\mathscr{X}$ of infinite block sequences can be extended to a witness of $\mathfrak{hom}_H$ with cardinality $\mathfrak{hom}_H$. Thus all of the requirements for $\mathfrak{hom}_H$ to be a very tame cardinal invariant are satisfied, and we are done. ∎

The following lemma is essentially just [28, Theorem 6.1.16], appropriately reformulated so that it is readily applicable in our context. The "proof" of the lemma just consists of linking together the necessary results and definitions from [28], for the benefit of those readers that are interested in chasing the lemma all the way to first principles.

LEMMA 2.5. *If $\mathfrak{x}$ is a very tame cardinal invariant such that $\mathfrak{x} < \mathfrak{c}$ in some forcing extension, then the iterated Sacks model satisfies $\omega_1 = \mathfrak{x} < \mathfrak{c}$* ([9]).

([9]) In [27], a *tame* cardinal invariant is defined to be one whose definition is of the form

$$\min\{|Y| \mid Y \subseteq \mathbb{R} \wedge \phi(Y) \wedge \psi(Y)\},$$

where the quantifiers of ϕ are restricted to the set Y or to ω, and $\psi(Y)$ is a sentence of the form $(\forall x \in \mathbb{R})(\exists y \in Y)(\theta(x, y))$, with θ a formula whose quantifiers range over natural and real numbers only, without mentioning the set Y (both formulas can have a real parameter). In that same paper [27, Theorem 0.2] it is established that, if there exists a proper class of measurable Woodin cardinals, then every tame cardinal invariant that can be made $< \mathfrak{c}$ in some forcing extension will be $< \mathfrak{c}$ in the iterated Sacks extension. So for the reader who feels entirely comfortable with large cardinal assumptions, much work can be saved by simply verifying that $\mathfrak{hom}_H$ satisfies the definition of a tame cardinal invariant (much simpler than that of a very tame invariant) and then appealing to the aforementioned theorem to conclude that $\mathfrak{hom}_H < \mathfrak{c}$ in the iterated Sacks model.

Proof. In [28, Theorem 6.1.16] it is stated that whenever I is an iterable Π^1_1 on Σ^1_1 σ-ideal on some Polish space, and $\mathfrak{x}$ is a very tame cardinal invariant, if $\mathfrak{x} < \mathrm{cov}^*(I)$ in some inner model of ZFC containing all ordinals or in a generic extension of such an inner model, then $\omega_1 = \mathfrak{x} < \mathrm{cov}^*(I)$ in every generic extension of every inner model of ZFC containing all ordinals, provided that such an extension satisfies CPA(I). Since Sacks forcing is forcing equivalent to the quotient of all Borel subsets of 2^ω modulo the ideal I of countable sets, and $\mathrm{cov}^*(I) = \mathfrak{c}$ (as $\mathrm{cov}^*(I)$ is defined to be the least number of elements of I required to cover an I-positive Borel set), it follows that whenever a very tame cardinal invariant $\mathfrak{x}$ satisfies $\mathfrak{x} < \mathfrak{c}$ in some forcing extension, necessarily $\omega_1 = \mathfrak{x} < \mathfrak{c}$ in the iterated Sacks model, as long as this model satisfies CPA(I) and the ideal I is iterable Π^1_1 on Σ^1_1. By [28, Proposition 6.1.3], under the hypothesized conditions, CPA(I) will hold in the generic extension obtained by CS iteration of the forcing $\mathbb{P}_I$, which is just the iterated Sacks model. Hence it only remains to see that the ideal I of countable sets is iterable Π^1_1 on Σ^1_1. The latter property follows directly from [16, 29.19, p. 231] (see [28, Definition 3.8.1] for the definition of Π^1_1 on Σ^1_1). As for iterable, the reader is urged to look at [28, Definition 5.1.3], which has three clauses, the first of which is satisfied by every Π^1_1 on Σ^1_1 ideal by [28, Proposition 2.1.22]. Now, that our ideal I of countable sets satisfies the second clause (defined by [28, Definition 3.9.21]) follows from [16, Exercise 14.13, p. 88]; and it satisfies the third clause as well by [28, Proposition 2.2.2] and/or by [28, Example 2.2.5]. ∎

THEOREM 2.6. *In the iterated Sacks model, the equality $\mathfrak{hom}_H = \omega_1$ holds.*

Proof. Refer to one of the models from [10, Section 4.3, Theorems 4.20 and 4.23] (obtained by means of "short" finite support iterations of c.c.c. forcing notions) that satisfy $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$ together with the existence of union ultrafilters. It is pointed out in the construction of such models (as part of the argument to conclude that $\mathrm{cov}(\mathcal{M})$ is small in these models) that the relevant union ultrafilter has a basis of $\mathrm{cov}(\mathcal{M})$-many sets of the form FU(X), and hence such a basis constitutes a witness for $\mathfrak{hom}_H < \mathfrak{c}$. Hence by Lemmas 2.4 and 2.5, the conclusion is that $\mathfrak{hom}_H = \omega_1$ in the iterated Sacks model. ∎

THEOREM 2.7. *The parametrized diamond principle $\diamondsuit(\mathfrak{hom}_H)$ holds in the iterated Sacks model.*

Proof. By Theorem 2.6, the cardinal invariant $\mathfrak{hom}_H$ equals ω_1 in the iterated Sacks model. Since this model is obtained by a suitable forcing iteration, Theorem 2.2 applies, and we conclude that $\diamondsuit(\mathfrak{hom}_H)$ holds in this model. ∎

3. Union ultrafilters from $\Diamond(\mathfrak{hom}_H)$. In the previous section we showed that $\mathfrak{hom}_H = \omega_1$ in the iterated Sacks model. Thus, by Theorem 2.2, $\Diamond(\mathfrak{hom}_H)$ holds in this model, so our proof that there are union ultrafilters in the iterated Sacks model will be complete once we prove that such an existential statement follows from $\Diamond(\mathfrak{hom}_H)$.

For the convenience of the reader, we state again the combinatorial principle $\Diamond(\mathfrak{hom}_H)$: for every Borel $F : 2^{<\omega_1} \to \wp([\omega]^{<\omega})$, there exists an ω_1-sequence g of block sequences such that, for every $f : \omega_1 \to 2$, $F(f \restriction \alpha)$ is almost FU-reaped by $g(\alpha)$ (i.e. $\mathrm{FU}(g(\alpha))$ reaps $F(f \restriction \alpha)$) for stationarily many $\alpha < \omega_1$.

In order to carry out our construction, we need a couple of lemmas.

LEMMA 3.1. *Given any block sequence $X \subseteq [\omega]^{<\omega}$, there exists an isomorphism $\varphi : \mathrm{FU}(X) \to [\omega]^{<\omega} \setminus \{\emptyset\}$ (a bijection such that, for any $x, y \in \mathrm{FU}(X)$, if $\max(x) < \min(y)$, then $\max(\varphi(x)) < \min(\varphi(y))$, and furthermore $\varphi(x \cup y) = \varphi(x) \cup \varphi(y)$).*

Proof. Since X is a block sequence, it is possible to enumerate its elements as $X = \{x_n \mid n < \omega\}$ in such a way that $\max(x_n) < \min(x_{n+1})$ for every $n < \omega$. Now just notice that the mapping $\bigcup_{n \in a} x_n \mapsto a$ is as required. ∎

LEMMA 3.2. *Given any sequence $\langle X_n \mid n < \omega \rangle$ of block sequences satisfying $(\forall n < \omega)(X_{n+1} \subseteq^* \mathrm{FU}(X_n))$, it is possible to choose, in a Borel way, a block sequence X such that $X \subseteq^* \mathrm{FU}(X)$ for all $n < \omega$.*

Proof. We will recursively construct $X = \{x_n \mid n < \omega\}$ in such a way that, for all $n < \omega$, $\max(x_n) < \min(x_{n+1})$, and furthermore $\mathrm{FU}(\{x_k \mid k \geq n\}) \subseteq X_n$. We do this as follows: Since each X_n is a block sequence, we can enumerate it as $X_n = \{x(n, k) \mid k < \omega\}$ in such a way that $\max(x(n, k)) < \min(x(n, k+1))$. Let $x_0 = x(0, 0)$ and, assuming that we know x_n, let k_n be (the least number) large enough that $\max(x_n) < \min(x(n+1, k_n))$ and $(\forall k \geq k_n)(\forall i \leq n)(x(n+1, k) \in \mathrm{FU}(X_i))$. Then just let $x_{n+1} = x(n+1, k_n)$. Then our sequence X, by construction, is a block sequence and satisfies $(\forall n < \omega)(\{x_k \mid k \geq n\} \subseteq \mathrm{FU}(X_n))$. ∎

These two lemmas encapsulate the key steps in the proof of the main theorem of this section. Before stating the theorem, we give a couple of definitions.

DEFINITION 3.3. Let u be a union ultrafilter.

- u is said to be *stable* if for every countable collection $\{X_n \mid n < \omega\}$ of pairwise disjoint families satisfying $(\forall n < \omega)(\mathrm{FU}(X_n) \in u)$, there exists a pairwise disjoint X such that $\mathrm{FU}(X) \in u$ and $(\forall n < \omega)(X \subseteq^* \mathrm{FU}(X))$.
- u is said to be an *ordered-union* ultrafilter if it satisfies the defining condition for union ultrafilters, where the "pairwise disjoint" requirement has

been strengthened to "ordered". That is, u is an ordered-union ultrafilter if $(\forall A \in u)(\exists X \text{ ordered})(A \supseteq \mathrm{FU}(X) \in u)$.

Thus, the property of being a stable ordered-union ultrafilter is stronger than simply being a union ultrafilter. It has been shown by Blass and Hindman [6, Theorem 4] that the existence of union ultrafilters that are not ordered is consistent with ZFC (in fact, it follows from Martin's Axiom, even when restricted to countable forcing notions, i.e. $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$). On the other hand, it is still open whether one can have (consistently) union ultrafilters that are not stable. We see below that the principle $\Diamond(\mathfrak{hom}_H)$ implies not only the existence of union ultrafilters, but in fact of stable ordered-union ultrafilters.

THEOREM 3.4. *The combinatorial principle $\Diamond(\mathfrak{hom}_H)$ implies the existence of stable ordered-union ultrafilters of character ω_1.*

Proof. We let sequences in $2^{<\omega_1}$ code pairs $\langle A, \vec{X} \rangle$, where $A \subseteq [\omega]^{<\omega}$, and $\vec{X} = \langle X_\xi \mid \xi < \alpha \rangle$ is a sequence of block sequences satisfying $X_\beta \subseteq^* \mathrm{FU}(X_\xi)$ for every $\xi < \beta < \alpha$. Also for every limit $\alpha < \omega_1$ we fix an increasing cofinal sequence $\langle \alpha_n \mid n < \omega \rangle$. Given a pair $\langle A, \vec{X} \rangle$, we define $F(\langle A, \vec{X} \rangle)$ as follows: We first pick, using Lemma 3.2, a block sequence Y such that $Y \subseteq^* \mathrm{FU}(X_{\alpha_n})$ for every $n < \omega$. We then let $\varphi_Y : \mathrm{FU}(Y) \to [\omega]^{<\omega} \setminus \{\emptyset\}$ be the isomorphism as in Lemma 3.1, and finally define $F(\langle A, \vec{X} \rangle) = \mathrm{ran}(\varphi_Y {\restriction} (A \cap \mathrm{FU}(Y)))$.

Now by our assumption that $\Diamond(\mathfrak{hom}_H)$ holds, we can obtain a $g : \omega_1 \to \{X \subseteq [\omega]^{<\omega} \mid X \text{ is an infinite block sequence}\}$ such that, for each $f : \omega_1 \to 2$, the set $\{\alpha < \omega_1 \mid F(f {\restriction} \alpha) \text{ is FU-reaped by } g(\alpha)\}$ is stationary. We use g to recursively define our ultrafilter as follows. Suppose that we already have $\langle X_\xi \mid \xi < \alpha \rangle$ satisfying $X_\eta \subseteq^* \mathrm{FU}(X_\xi)$ whenever $\xi < \eta < \alpha$. Now take the same cofinal sequence $\langle \alpha_n \mid n < \omega \rangle$ as in the construction of F, and let Y be as in Lemma 3.2 for the sequence $\langle X_{\alpha_n} \mid n < \omega \rangle$. Let $\varphi_Y : \mathrm{FU}(Y) \to [\omega]^{<\omega} \setminus \{\emptyset\}$ be the isomorphism from Lemma 3.1, and define $X_\alpha = \varphi_Y^{-1}[g(\alpha)] \subseteq Y$. Then by construction, the final sequence $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ will be such that $\{\mathrm{FU}(X_\alpha) \mid \alpha < \omega_1\}$ generates a filter, which will be a union ultrafilter provided that we can show that it is actually an ultrafilter.

To see this, let $A \subseteq [\omega]^{<\omega}$ be arbitrary. Let $f : \omega_1 \to 2$ be the function that codes the branch $\langle A, \langle X_\alpha \mid \alpha < \omega_1 \rangle \rangle$ of our tree. By the assumption on g, there will be an α (in fact, stationarily many of them, although we really only need one) such that $F(\langle A, \langle X_\xi \mid \xi < \alpha \rangle \rangle)$ is FU-reaped by $g(\alpha)$. But recall that the former set is just $\mathrm{ran}(\varphi_Y {\restriction} (A \cap \mathrm{FU}(Y)))$, where Y was obtained from $\langle X_{\alpha_n} \mid n < \omega \rangle$ by means of Lemma 3.2, and φ_Y is the isomorphism from Lemma 3.1. Since $\mathrm{FU}(g(\alpha))$ reaps this set, by pulling everything back via the isomorphism φ_Y, we find that $\varphi_Y^{-1}[\mathrm{FU}(g(\alpha))] = \mathrm{FU}(\varphi^{-1}[g(\alpha)]) = \mathrm{FU}(X_\alpha)$ reaps $\varphi_Y^{-1}[\mathrm{ran}(\varphi_Y {\restriction} (A \cap \mathrm{FU}(Y)))] = A \cap \mathrm{FU}(Y)$,

and therefore also A. Thus our filter is actually an ultrafilter, and in fact a stable ordered-union ultrafilter. ∎

As a corollary of the above, we obtain an answer to the question posed in [10, Question 4.11, p. 106], which was the main motivation behind this work.

COROLLARY 3.5. *In the iterated Sacks model, the statement "there exists a stable ordered-union ultrafilter of character ω_1" holds.*

Proof. By Theorem 2.7, the combinatorial principle $\Diamond(\mathfrak{hom}_H)$ holds in the iterated Sacks model, thus by Theorem 3.4 this implies that there exists an ultrafilter as required in this model. ∎

REFERENCES

[1] A. Arhangelski, *Groupes topologiques extrémalement discontinus*, C. R. Acad. Sci. Paris 265 (1967), 822–825.
[2] J. Baumgartner, *A short proof of Hindman's theorem*, J. Combin. Theory Ser. A 17 (1974), 384–386.
[3] A. Blass, *Combinatorial Cardinal Characteristics of the Continuum*, in: Handbook of Set Theory, Vol. 1, M. Foreman and A. Kanamori (eds.), Springer, 2010, 395–489.
[4] A. Blass, *Ultrafilters related to Hindman's finite-unions theorem and its extensions*, in: Logic and Combinatorics, S. Simpson (ed.), Contemp. Math. 65, Amer. Math. Soc., 1987, 89–124.
[5] A. Blass, *Homogeneous sets from several ultrafilters*, Topology Appl. 156 (2009), 2581–2594.
[6] A. Blass and N. Hindman, *On strongly summable ultrafilters and union ultrafilters*, Trans. Amer. Math. Soc. 304 (1987), 83–99.
[7] A. Blass and S. Shelah, *Near coherence of filters III: A simplified consistency proof*, Notre Dame J. Formal Logic 30 (1989), 530–538.
[8] D. Chodounský and O. Guzmán, *There are no P-points in Silver extensions*, arXiv: 1703.02082 (2017).
[9] T. Eisworth, *Forcing and stable-ordered union ultrafilters*, J. Symbolic Logic 67 (2002), 449–464.
[10] D. Fernández Bretón, *Strongly summable ultrafilters: some properties and generalizations*, PhD thesis, York Univ, Toronto, Ont., 2015.
[11] D. Fernández Bretón, *Strongly summable ultrafilters, union ultrafilters, and the trivial sums property*, Canad. J. Math. 68 (2016), 44–66.
[12] N. Hindman, *The existence of certain ultra-filters on N and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc. 36 (1972), 341–346.

[13] N. Hindman, *Summable ultrafilters and finite sums*, in: Logic and Combinatorics, S. Simpson (ed.), Contemp. Math. 65, Amer. Math. Soc., 1987, 263–274.

[14] N. Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combin. Theory Ser. A 17 (1974), 1–11.

[15] N. Hindman, J. Steprāns and D. Strauss, *Semigroups in which all strongly summable ultrafilters are sparse*, New York J. Math. 18 (2012), 835–848.

[16] A. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, 1994.

[17] P. Krautzberger, *On union ultrafilters*, Order 29 (2012), 317–343.

[18] K. Kunen, *Some points in βN*, Math. Proc. Cambridge Philos. Soc. 80 (1976), 385–398.

[19] V. I. Malykhin, *On extremely disconnected topological groups*, Uspekhi Mat. Nauk 34 (1979), no. 6, 59–66 (in Russian).

[20] P. Matet, *Some filters of partitions*, J. Symbolic Logic 53 (1988), 540–553.

[21] A. Miller, *There are no Q-points in Laver's model for the Borel conjecture*, Proc. Amer. Math. Soc. 78 (1980), 103–106.

[22] J. T. Moore, M. Hrušák and M. Džamonja, *Parametrized ◇-principles*, Trans. Amer. Math. Soc. 356 (2004), 2281–2306.

[23] I. V. Protasov, *Ultrafilters and topologies on groups*, Siberian Math. J. 34 (1993), 938–952 (translated from Sibirsk. Mat. Zh. 34 (1993), no. 5, 163–180).

[24] U. A. Ramos-García, *Extremal disconnectedness and ultrafilters*, preprint, 2017.

[25] E. Reznichenko and O. Sipacheva, *Discrete subsets in topological groups and countable extremally disconnected groups*, arXiv:1608.03546v3 (2017).

[26] E. Wimmers, *The Shelah P-point independence theorem*, Israel J. Math. 43 (1982), 28–48.

[27] J. Zapletal, *Isolating cardinal invariants*, J. Math. Logic 3 (2003), 146–162.

[28] J. Zapletal, *Forcing Idealized*, Cambridge Tracts in Math. 174, Cambridge Univ. Press, 2008.

[29] Y. Y. Zheng, *Selective ultrafilters on FIN*, Proc. Amer. Math. Soc. 145 (2017), 5071–5086.

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