$\mathbb{Z}$-RAMSEY ULTRAFILTERS

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Abstract. We study $\mathbb{Z}$-Ramsey ultrafilters, ultrafilters containing witnesses to every shift-invariant instance of Ramsey's theorem. We prove that it is consistent that there are no $\mathbb{Z}$-Ramsey ultrafilters. We also prove that every $(\mathbb{Z}, 3)$-Ramsey ultrafilter, as well as every $\mathbb{Z}$-Ramsey P-point, is selective. Further, we exhibit a generic extension—using quotient algebras of the form $\mathcal{P}(\mathbb{Z})/\mathcal{I}$ for certain F_σ-ideals—that contains P-points that are not $\mathbb{Z}$-Ramsey ultrafilters, thereby addressing open questions raised by Petrenko and Protasov [10, 11].

1. Introduction

The main object of study in this paper are $\mathbb{Z}$-Ramsey ultrafilters: ultrafilters containing homogeneous sets for every colouring that is invariant with respect to the usual shift action of $\mathbb{Z}$ on itself. These ultrafilters were first introduced by Petrenko and Protasov [10], in the more general context of a G-Ramsey ultrafilter over any G-set X (defined as ultrafilters on X containing c-homogeneous sets for every colouring c of $[X]^2$ that is invariant under the action of G on X). They also introduced G-selective ultrafilters (ultrafilters with the property that for each G-invariant partition of X, either one of the pieces of the partition belongs to the ultrafilter, or the ultrafilter contains a selector for the partition); in a related paper [11] (published two years later but probably written concurrently), the same two authors proved that, in general, any G-Ramsey ultrafilter is G-selective [11, Theorem 5.3] but the reverse implication does not hold (because, e.g., G-selective ultrafilters can always be proven to exist in ZFC [11, Theorem 5.1], whereas the same cannot be said of G-Ramsey ultrafilters, as we will discuss in what follows).

So, more concretely, we will say that an ultrafilter u on $\mathbb{Z}$ is $(\mathbb{Z}, k)$**-Ramsey** if, for every colouring $c : [\mathbb{Z}]^k \longrightarrow 2$ that is invariant under the shift action of $\mathbb{Z}$ (that is, with the property that $c(\{x_1, \dots, x_k\}) = c(\{x_1 + z, \dots, x_k + z\})$ for every $z \in \mathbb{Z}$), the ultrafilter contains a c-homogeneous set. We will simply say that an ultrafilter is $\mathbb{Z}$-Ramsey when it is $(\mathbb{Z}, 2)$-Ramsey. One could say that, in the same way that selective ultrafilters are related to Ramsey's theorem, $\mathbb{Z}$-Ramsey ultrafilters are related to the $\mathbb{Z}$-Ramsey theorem from [1]. The paper [11] contains a plethora of results regarding $\mathbb{Z}$-Ramsey ultrafilters (particularly in terms of necessary conditions, as well as a number of properties of ultrafilters that imply

Date: June 8, 2026.

2010 *Mathematics Subject Classification.* Primary 03E35; Secondary 03E05, 05D10, 54D80.

Key words and phrases. $\mathbb{Z}$-Ramsey ultrafilter, selective ultrafilter, P-point, F_σ-ideal, $\mathbb{Z}$-Ramsey theorem, central set.

The first author's research has been supported by Secihti, scholarship 298332, and by PAPIIT grant IN104419. The research of the second author was partially supported by grants SIP-20253559 and SIP-20260817 from IPN, as well as by grant CBF2023-2024-334 from Secihti. The third named author was partially supported by PAPIIT grants IN104419 and IN116225.

that they are *not* $\mathbb{Z}$-Ramsey ultrafilters). Of particular importance are certain theorems concerning selective ultrafilters, such as [10, Theorem 2.6], stating that every $\mathbb{Z}$-Ramsey ultrafilter containing a 1-thin set (to be defined below) must be selective; in a similar vein, [11, Corollary 2.7] establishes that every $\mathbb{Z}$-Ramsey ultrafilter that is also a Q-point, must necessarily be selective. Notably, it is also the case that $(\mathbb{Z},4)$-Ramsey ultrafilters must be selective [11, Corollary 2.8]; in this paper we improve that result to $(\mathbb{Z},3)$-ultrafilters (see Theorem 5.1 below). It is not clear yet that this number 3 can be further improved: one of the main questions motivating the work on this paper is [10, Question 5.1] whether every $\mathbb{Z}$-Ramsey ultrafilter must be selective. In this paper we provide a partial answer to this question, given by a theorem that is reminiscent to several of the results mentioned in this paragraph: Theorem 5.6 below establishes that every $\mathbb{Z}$-Ramsey ultrafilter that is also a P-point must be selective.

Another related question, mentioned in [11, p. 458] (see the short paragraph right after [11, Corollary 2.7]) is whether every P-point in $\mathbb{Z}^*$ is a $\mathbb{Z}$-Ramsey ultrafilter. We also address this question in the present paper, answering it in the negative by exhibiting a family of forcing notions (generated by certain F_σ-ideals) with the property that they provide a generic extension containing P-points (moreover, P-points all of whose elements are positive with respect to the relevant ideal) that are not $\mathbb{Z}$-Ramsey ultrafilters. In the process of developing this proof, we provide a negative Ramsey-theoretic result by showing that there exist certain sets carrying $\mathbb{Z}$-invariant colourings without "large" monochromatic subsets (with "large" interpreted as positive with respect to the ideal at hand). In the end, although not directly relevant to the ultrafilter questions, we complement the negative Ramsey-theoretic result just mentioned with a positive one, stating that for each $\mathbb{Z}$-invariant colouring of $[\mathbb{Z}]^2$, it is possible to find a "large" (in a sense that will be made precise) monochromatic set, see Theorem 6.2 below.

In Section 2, we summarize some preliminary results and definitions that will be used throughout the paper. In Section 3, we proceed to study some ideals, in particular what we call F_σ similarity invariant ideals and, for some particular cases of these ideals, we prove that they do not satisfy the corresponding version of Ramsey's theorem; as a corollary of this, we conclude that it is possible to force the existence of P-points that are not $\mathbb{Z}$-Ramsey ultrafilters. In section 4 we establish that the existence of $\mathbb{Z}$-Ramsey ultrafilters is independent from ZFC, and, in Section 5, we show that $\mathbb{Z}$-Ramsey ultrafilters that are also P-points must be selective. Finally, Section 6 contains a Ramsey-theoretic statement about the ideal generated by 1-thin sets, and states a related open question.

2. Preliminaries

Let us recall the basic notions we will use throughout the paper.

A non-empty family $\mathcal{I}$ of subsets of a set X is an **ideal** on X if it is closed under taking subsets and finite unions and does not contain the set X. In this paper we assume that all ideals on X contain all finite subsets of X. We say that an ideal $\mathcal{I}$ on X is **tall** if for each $Y \in [X]^\omega$ there exists $I \in \mathcal{I}$ such that $I \cap Y$ is infinite. Given an ideal $\mathcal{I}$ on a set X, we denote by $\mathcal{I}^+$ the family of $\mathcal{I}$-positive sets, that is, subsets of X which are not in $\mathcal{I}$. If $\mathcal{I}$ is an ideal on X and $Y \in \mathcal{I}^+$, we denote by $\mathcal{I} \upharpoonright Y$ the ideal $\{I \cap Y \mid I \in \mathcal{I}\}$ on Y. When X is countable, an ideal $\mathcal{I}$ on X can be regarded as an ideal on ω via any bijection between X and ω, so we will assume our ideals to be ideals on ω.

We consider $\mathcal{P}(\omega)$ equipped with the natural topology induced by identifying each subset of ω with its characteristic function, where 2^ω is given the product topology. We call an ideal $\mathcal{I}$ a Borel (analytic, co-analytic,...) ideal on ω if $\mathcal{I}$ is an ideal on ω and $\mathcal{I}$ is Borel (analytic, co-analytic,...) in this topology.

There is an extremely close and useful connection between F_σ ideals and lower semicontinuous submeasures. A **submeasure on** a set X is a function $\varphi\colon \mathcal{P}(X) \to [0,\infty]$ satisfying $\varphi(\emptyset) = 0$, if $A \subseteq B$ then $\varphi(A) \leqslant \varphi(B)$, and $\varphi(A \cup B) \leqslant \varphi(A) + \varphi(B)$. To avoid trivialities, we also require that $\varphi(F) < \infty$ for all finite subset F of X. If φ is a submeasure on ω and satisfies $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for all $A \subseteq \omega$ then φ is called a **lower semicontinuous submeasure**, abbreviated by lscsm. To each lscsm φ on ω naturally correspond the ideal $\mathsf{Fin}(\varphi) = \{A \subseteq \omega \mid \varphi(A) < \infty\}$. It is immediate from the definition that $\mathsf{Fin}(\varphi)$ is an F_σ ideal. The following fundamental theorem of Mazur is key to the study of F_σ-ideals.

Theorem 2.1 ([9]). *Let $\mathcal{I}$ be an ideal on ω. Then $\mathcal{I}$ is an F_σ ideal if and only if there is a lscsm φ such that $\mathcal{I} = \mathsf{Fin}(\varphi)$.* □

A useful characterization of selective ultrafilters is due to Mathias:

Theorem 2.2 ([8]). *Let u be an ultrafilter on ω. Then, u is selective if and only if $u \cap \mathcal{I} \neq \emptyset$ for every analytic tall ideal $\mathcal{I}$ on ω.* □

Various types of ultrafilters can be added generically by the quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is some definable ideal (usually one such that $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals; for example, F_σ-ideals). Indeed, using Theorem 2.1, it is easy to prove the following result, which was first observed in [7]:

Remark 2.3 ([7]). If $\mathcal{I}$ is an F_σ ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed; in fact, $\mathcal{I}$ is a P^+-ideal. □

An ideal $\mathcal{I}$ on ω is a **$\mathbf{P}^+$-ideal** if for every decreasing sequence $\{X_n \mid n < \omega\}$ of $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive set X such that $X \subseteq^* X_n$, for all $n < \omega$.

The aforementioned quotient algebra provides a straightforward method for the consistent construction of various types of ultrafilters [6]. Instead of dealing with the quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$, it is common to use the non-separative equivalent forcing notion $(\mathcal{I}^+, \subseteq)$.

Following [5], we will be interested in the following G-invariant Ramsey type properties of ideals over a group G: Let $\mathcal{I}$ be an ideal on a countably infinite group G. We say that $\mathcal{I}$ satisfies

$$G \xrightarrow{\text{left-inv.}} (\mathcal{I}^+)^2_2$$

if for every colouring $c\colon [G]^2 \longrightarrow 2$ that is invariant under the left multiplication action of G (that is, with the property that $c(\{x_1, x_2\}) = c(\{gx_1, gx_2\})$ for every $g \in G$), there is an $\mathcal{I}$-positive set X homogeneous with respect to c. We say that $\mathcal{I}$ satisfies

$$\mathcal{I}^+ \xrightarrow{\text{left-inv.}} (\mathcal{I}^+)^2_2$$

if for every $\mathcal{I}$-positive set X and every left-invariant colouring $c\colon [G]^2 \longrightarrow 2$ there is an $\mathcal{I}$-positive subset $Y \subseteq X$ homogeneous with respect to c. Thus, we shall call an ideal $\mathcal{I}$ **left-invariant Ramsey**(G), if $G \xrightarrow{G\text{-inv.}} (\mathcal{I}^+)^2_2$, and we shall say that

$\mathcal{I}$ is **left-invariant Ramsey** if $\mathcal{I}^+ \xrightarrow{G\text{-inv.}} (\mathcal{I}^+)^2_2$. Let X be an $\mathcal{I}$-positive set. A partition $\{F_n \mid n < \omega\}$ of X into finite sets is called **thin** if, for every $n > 0$ and $F = \bigcup_{k<n} F_k$, the following conditions are satisfied: $D(F_n, F_n) \cap D(F, F) = \emptyset$, $D(F_n, F_n) \cap D(F_n, F) = \emptyset$, and $D(F_n, F) \cap D(F, F) = \emptyset$, where D is defined as follows.

Definition 2.4. Given a group G and subsets $A, B \subseteq G$, we denote $D(A, B) = \{y^{-1}x \mid x \in A,\, y \in B \text{ and } x \neq y\}$. When $A = B$, we simply write $D(A)$.

The following definition delineates the main object of study in this paper.

Definition 2.5. Given an $n \in \mathbb{N}$, an ultrafilter $u \in \beta\mathbb{Z}$ is called $(\mathbb{Z}, n)$**-Ramsey** if, for every colouring $c : [\mathbb{Z}]^n \longrightarrow 2$ that is $\mathbb{Z}$-invariant, there exists an $A \in u$ such that $[A]^n$ is c-monochromatic. In case $n = 2$, we will simply say that u is a $\mathbb{Z}$**-Ramsey ultrafilter**.

Note that every ultrafilter is $(\mathbb{Z}, 1)$-Ramsey. By the usual arguments involving the dropping of one coordinate of a tuple, every $(\mathbb{Z}, n+1)$-Ramsey ultrafilter is seen to be a $(\mathbb{Z}, n)$-Ramsey ultrafilter. Petrenko and Protasov in [11] proved that every $(\mathbb{Z}, 4)$-Ramsey ultrafilter must be selective.

3. A generic P-point not $\mathbb{Z}$-Ramsey

In this section, we develop the necessary tools to negatively answer the question mentioned in [11, p. 458], showing that for certain F_σ-ideals $\mathcal{I}$ on $\mathbb{Z}$, the quotient algebra $\mathcal{P}(\mathbb{Z})/\mathcal{I}$ generically adds an ultrafilter u which is a P-point but not $\mathbb{Z}$-Ramsey.

Recall that a subset A of a countably infinite group G is k**-thin** ($k \geq 1$) if

$$|gA \cap A| \leq k$$

for each $g \in G \setminus \{e_G\}$ (see [11, 12]). In this context, we define the natural ideals generated by such sets as follows:

$$\mathcal{I}_{k\text{-thin}} = \left\{ I \subset G \mid (\exists \{A_i \mid i < n\} \subset k\text{-thin}) \left(I \subseteq \bigcup_{i<n} A_i \right) \right\},$$

where k-thin $= \{A \subset G \mid A \text{ is a } k\text{-thin set}\}$. The family k-thin is hereditary; that is, for each $A \in k$-thin and each $B \subseteq A$, it follows that $B \in k$-thin. Also, note that

$$\mathcal{I}_{k\text{-thin}} \subseteq \mathcal{I}_{(k+1)\text{-thin}}$$

for every $k \geq 1$.

Lemma 3.1. *Let G be a countably infinite group. Then $\mathcal{I}_{k\text{-thin}}$ is a tall F_σ ideal.*

Proof. To see that $\mathcal{I}_{k\text{-thin}}$ is tall, it suffices to prove that $\mathcal{I}_{1\text{-thin}}$ is tall. Let $X \subseteq G$ be an infinite set. Recursively choose, for each $n < \omega$,

$$y_n \in X \setminus \left(\{y_i y_j^{-1} y_k \mid i, j, k < n\} \cup \{y_i \mid i < n\} \right),$$

and let $Y = \{y_n \mid n < \omega\}$. By construction, $Y \subseteq X$ is an infinite set; we claim that this set is 1-thin. To see this, let $g \in G \setminus \{e_G\}$ be arbitrary and assume, to get a contradiction, that $|gY \cap Y| \geq 2$. This means there are two distinct i, j such that $y_i, y_j \in gY$, so there are l, k such that $y_i = gy_k$ and $y_j = gy_l$. Note that this necessarily implies that i, j, k, l are pairwise distinct, and $g = y_i y_k^{-1} = y_j y_l^{-1}$. By taking inverses on both sides of the last equation if necessary, we may assume that

$i = \max\{i, j, k, l\}$, and from here we may deduce $y_i = y_j y_l^{-1} y_k$, contrary to the recursive definition of the y_n.

On the other hand, as the function $\cup\colon \mathcal{P}(G) \to \mathcal{P}(G)$ given by

$$\cup(A, B) = A \cup B$$

is continuous and the family k-thin is hereditary, to check that $\mathcal{I}_{k\text{-}\mathsf{thin}}$ is F_σ, it suffices to show that k-thin is closed in $\mathcal{P}(G)$. Take any $A \in \mathcal{P}(G) \setminus k$-$\mathsf{thin}$. Then there exists a $g \in G \setminus \{e_G\}$ such that $|gA \cap A| > k$. Pick $a_0, a_1, \ldots, a_k \in A$ such that $ga_0, ga_1, \ldots, ga_k \in A$. Observe that $U = \{X \in \mathcal{P}(G) \mid \{a_i \mid i \leq k\} \cup \{ga_i \mid i \leq k\} \subseteq X\}$ is open, $U \cap k$-$\mathsf{thin} = \emptyset$, and $A \in U$. Therefore, k-thin is a closed set. □

Since the family k-thin is hereditary and closed in $\mathcal{P}(G)$, it follows from Theorem 2.1 that we can provide a witness lscsm, $\varphi_{k\text{-}\mathsf{thin}}$, showing that the ideal $\mathcal{I}_{k\text{-}\mathsf{thin}}$ is an F_σ ideal. To this end, we define $F_0 = k$-thin and $F_n = \{A \cup B \mid A, B \in F_{n-1}\}$ for $n \geq 1$. Thus, $F_n \subseteq F_{n+1}$ $(n < \omega)$ and $\mathcal{I}_{k\text{-}\mathsf{thin}} = \bigcup_{n<\omega} F_n$. For any $E \in [G]^{<\omega}$, we set

$$\varphi_{k\text{-}\mathsf{thin}}(E) = \min\{n + 1 \mid E \in F_n\},$$

and define $\varphi_{k\text{-}\mathsf{thin}}(A) := \lim_{n\to\infty} \varphi_{k\text{-}\mathsf{thin}}(A \cap \{g_i \mid i < n\})$ for any $A \in \mathcal{P}(G)$, where $\{g_n \mid n < \omega\}$ is a fixed enumeration of G. Moreover, since k-thin is left-invariant (meaning $gA \in k$-thin for all $A \in k$-thin and $g \in G$), it follows that both $\varphi_{k\text{-}\mathsf{thin}}$ and the ideal $\mathcal{I}_{k\text{-}\mathsf{thin}}$ are left-invariant.

The 1-thin sets play a crucial role in determining when a $\mathbb{Z}$-Ramsey ultrafilter is selective, as shown by the following result.

Theorem 3.2 ([10], Theorem 2.6)**.** *Let u be a $\mathbb{Z}$-Ramsey ultrafilter containing a* 1-*thin set. Then, u must be selective.* □

On the other hand, Mathias's result (Theorem 2.2) tells us that containing a 1-thin set is, in fact, a necessary condition for a $\mathbb{Z}$-Ramsey ultrafilter to be selective. Thus, a $\mathbb{Z}$-Ramsey ultrafilter u is selective if and only if $u \cap \mathcal{I}_{1\text{-}\mathsf{thin}} \neq \emptyset$.

Definition 3.3. Let G be a countably infinite group. An F_σ ideal $\mathcal{I}$ on G (witnessed by a lscsm φ) is called a **similarity-invariant ideal** if the following conditions hold:

(i) φ is left-invariant; that is, $\varphi(A) = \varphi(gA)$ for all $A \subseteq G$ and all $g \in G$.
(ii) For every $F \in [G]^{<\omega}$ and every $n < \omega$, there exists $E \in [G]^{<\omega}$ such that $\varphi(E) \geq n$, $E \cap F = \emptyset$, $D(E, E) \cap D(F, F) = \emptyset$, $D(E, E) \cap D(E, F) = \emptyset$ and $D(E, F) \cap D(F, F) = \emptyset$.

Examples of similarity-invariant ideals relevant to our purposes are the van der Waerden ideal $\mathcal{W}$ and the k-thin ideal $\mathcal{I}_{k\text{-}\mathsf{thin}}$ over the group of integers $\mathbb{Z}$. Recall that a set $A \subseteq \mathbb{Z}$ is called an **AP-set** if it contains arbitrarily long arithmetic progressions. The **van der Waerden ideal** is defined by

$$\mathcal{W} = \{A \subseteq \mathbb{Z} \mid A \text{ is not an AP-set}\}.$$

It is well known that $\mathcal{W}$ is a tall F_σ-ideal on $\mathbb{Z}$. Note that $\mathcal{W}$ is an invariant ideal and, since each $A \in k$-thin contains no arithmetic progressions of length $k + 2$, it follows that $\mathcal{I}_{k\text{-}\mathsf{thin}} \subseteq \mathcal{W}$, and hence $\mathcal{W}^+ \subseteq \mathcal{I}^+_{k\text{-}\mathsf{thin}}$.

Lemma 3.4. *If G is the group of integers $\mathbb{Z}$, then both $\mathcal{W}$ and $\mathcal{I}_{k\text{-}\mathsf{thin}}$ are similarity-invariant ideals.*

Proof. We only need to show that $\mathcal{W}$ satisfies clause (ii) from Definition 3.3. Let $F \in [\mathbb{Z}]^{<\omega}$ and $n < \omega$. Let $\varphi_{\mathcal{W}}$ be a lscsm witness that $\mathcal{W}$ is an F_σ-ideal. First, observe that $a + d\mathbb{N} = \{a + dx \mid x \in \mathbb{N}\} \in \mathcal{W}^+$ for all $a \in \mathbb{Z}$ and $d \in \mathbb{N}$. Take $a_F > \max\{|x| \mid x \in F\}$ and $d_F > \max\{a_F - x \mid x \in F\}$. Since $a_F + d_F\mathbb{N} \in \mathcal{W}^+$, we can find a finite subset E of $a_F + d_F\mathbb{N}$ with $\varphi_{\mathcal{W}}(E) \geq n$. Clearly, $E \cap F = \emptyset$. Since $a_F - x = (a_F - y) + (y - x) > y - x$ if $x, y \in F$ and $x < y$ and, $D(E, E) \subseteq \pm d_F\mathbb{N}$, it follows that $D(E, E) \cap D(F, F) = \emptyset$. To see that $D(E, E) \cap D(E, F) = \emptyset$, assume for contradiction that there exist $x \in F$ and $y \in \mathbb{N}$ with $a_F + d_F y \in E$ such that $(a_F + d_F y) - x = d_F z$ for some $z \in \mathbb{N}$. Then $a_F - x = d_F(z - y)$; this implies d_F is a divisor of $a_F - x$. In particular, $d_F \leq a_F - x$, which contradicts $d_F > \max\{a_F - x \mid x \in F\}$. Finally, as $z - x < (z - x) + (a_F - z) + d_F y = (a_F + d_F y) - x$ if $x, z \in F$ with $x < z$ and $y \in \mathbb{N}$, we conclude that $D(E, F) \cap D(F, F) = \emptyset$. □

Lemma 3.5. *Let $\mathcal{I}$ be an F_σ similarity-invariant ideal on a countably infinite group G. There exists an $\mathcal{I}$-positive set X, a thin partition $\{F_n \mid n < \omega\}$ of X consisting of finite sets and a left-invariant colouring $c\colon [G]^2 \longrightarrow 2$ such that every c-homogeneous subset Y of X is either finite or a partial selector of $\{F_n \mid n < \omega\}$ (that is, $|Y \cap F_n| \leq 1$ for all $n < \omega$).*

Proof. Let φ be a lscsm witnessing the fact that $\mathcal{I}$ is a similarity-invariant ideal. Using clause (ii) from Definition 3.3, we can recursively construct a family $\{F_n \mid n < \omega\}$ of finite subsets of G such that:

(1) $\varphi(F_n) \geq n$ for $n < \omega$.
(2) $F_n \cap F_m = \emptyset$ for $n < m < \omega$.
(3) If $n > 0$ and $F = \bigcup_{k<n} F_k$, then $D(F_n, F_n) \cap D(F, F) = \emptyset$, $D(F_n, F_n) \cap D(F_n, F) = \emptyset$, and $D(F_n, F) \cap D(F, F) = \emptyset$.

We set $X = \bigcup_{n<\omega} F_n$. Since $\varphi(F_n) \to \infty$, it follows that $X \in \mathcal{I}^+$. Define the colouring $c\colon [G]^2 \longrightarrow 2$ by $c(\{x, y\}) = 0$ if there exist $n < \omega$ and $z, w \in F_n$ such that $z^{-1}w = x^{-1}y$. Clearly, c is left-invariant. By clause (3), any c-homogeneous subset Y of X is either homogeneous in colour 0 and finite (contained within one single F_n), or infinite, in which case it is homogeneous in colour 1 and Y is a partial selector for the family $\{F_n \mid n < \omega\}$. □

We next demonstrate that when G is the group of integers $\mathbb{Z}$, both $\mathcal{W}$ and $\mathcal{I}_{k\text{-thin}}$ fail to be $\mathbb{Z}$-invariant Ramsey.

Theorem 3.6. *There exists a $\mathcal{W}$-positive set (resp., $\mathcal{I}_{k\text{-thin}}$-positive set) $X \subseteq \mathbb{Z}$ and a $\mathbb{Z}$-invariant colouring $c\colon [\mathbb{Z}]^2 \longrightarrow 2$ such that every c-homogeneous set $Y \subseteq X$ belongs to $\mathcal{W}$ (resp., $\mathcal{I}_{k\text{-thin}}$); that is,*

$$\mathcal{W}^+ \xrightarrow[\quad]{\textit{left-inv.}}\!\!\!\!\!\!\!\!\!\!/\quad\; (\mathcal{W}^+)^2_2 \left(\textit{resp., } \mathcal{I}^+_{k\text{-thin}} \xrightarrow[\quad]{\textit{left-inv.}}\!\!\!\!\!\!\!\!\!\!/\quad\; (\mathcal{I}^+_{k\text{-thin}})^2_2 \right).$$

Proof. According to Lemmas 3.4 and 3.5, there exists a $\mathcal{W}$-positive set X, a thin partition $\{F_n \mid n < \omega\}$ of X consisting of finite sets and a left-invariant colouring $c\colon [\mathbb{Z}]^2 \longrightarrow 2$ such that every c-homogeneous subset Y of X is either finite or a partial selector of $\{F_n \mid n < \omega\}$.

Claim 3.7. If Y is a partial selector of $\{F_n \mid n < \omega\}$, then $Y \in 1\text{-thin}$.

Proof of the claim. Let $x \in \mathbb{Z} \setminus \{0\}$ be arbitrary and assume, for the sake of contradiction, that $|(x+Y) \cap Y| \geq 2$. Then, there are two distinct elements $y_0, y_1 \in$

$(x+Y)\cap Y$. Thus, there exist $y_2, y_3 \in Y$ such that $y_0 = x+y_2$ and $y_1 = x+y_3$. Note that this implies that y_0, y_1, y_2, y_3 are pairwise distinct, and $x = y_0 - y_2 = y_1 - y_3$. By taking negatives on both sides of the last equation if necessary, we may assume that $y_0 \in F_n$, where $n = \max\{k \mid |F_k \cap \{y_i \mid i < 4\}| = 1\}$. Thus, since Y is a partial selector of $\{F_n \mid n < \omega\}$, we may deduce that $(F_n \cap F) \cap (F \cap F) \neq \emptyset$, where $F = \bigcup_{k<n} F_k$, contradicting the fact that the partition $\{F_n \mid n < \omega\}$ is thin. ■

As 1-thin $\subseteq \mathcal{I}_{k\text{-thin}} \subseteq \mathcal{W}$ for $k \geq 1$, the result follows. □

Corollary 3.8. *Let u be the $\mathcal{P}(\mathbb{Z})/\mathcal{I}_{k\text{-thin}}$-generic ultrafilter; then u is a P-point that is not $\mathbb{Z}$-Ramsey.*

Proof. By Observation 2.4 from [6], it follows that u is a P-point and, since the quotient algebra $\mathcal{P}(\mathbb{Z})/\mathcal{I}_{k\text{-thin}}$ does not add new reals, Theorem 3.6 implies that there exists a condition $X \in \mathcal{I}^+_{k\text{-thin}}$ such that X forces that $\dot{u}$ is not $\mathbb{Z}$-Ramsey. □

4. A $\mathbb{Z}$-Ramsey ultrafilter implies a P-point

The question whether the existence of a $\mathbb{Z}$-Ramsey ultrafilter is independent from ZFC is a very natural one. The arguments from [11, p. 459] show that the existence of a B-Ramsey ultrafilter, where B is a Boolean group (and the notion of B-Ramsey is defined analogously), imply the existence of a P-point; however, no such arguments have been made for $\mathbb{Z}$-Ramsey ultrafilters. In this section, we proceed to prove that, if there exists a $\mathbb{Z}$-Ramsey ultrafilter, then there is a P-point. In particular, it is consistent with ZFC that there are no $\mathbb{Z}$-Ramsey ultrafilters.

From now on, we will assume that our ultrafilters are over $\mathbb{N}$, the set of positive integers. We do not lose generality by doing so, since every ultrafilter over $\mathbb{Z}$ must contain either $\mathbb{N}$ or $-\mathbb{N}$, and the mapping $x \longmapsto -x$ is a bijection between these two sets that preserves addition. We now recall two key definitions.

Definition 4.1 (Petrenko and Protasov, [11], p. 456)**.** If u is an ultrafilter, then $D(u)$ is defined to be the filter generated by all sets of the form $D(A)$, with $A \in u$.

In general, $D(u)$ need not be an ultrafilter; the following theorem will be crucial for us.

Theorem 4.2 (Petrenko and Protasov, [11], Thm. 2.2 (ii))**.** *Let u be a free ultrafilter on $\mathbb{Z}$ such that $\mathbb{Z}^+ \in u$. Then, $D(u)$ is an ultrafilter if and only if u is $\mathbb{Z}$-Ramsey.* □

The other main tool we will use are the functions $\lambda, \mu : \mathbb{N} \longrightarrow \omega$. These are defined in the following way: given an $n \in \mathbb{N}$, and after expanding it in binary notation, $\lambda(n)$ is the position of the least non-zero digit and $\mu(n)$ is the position of the last one. Another equivalent definition is that $\lambda(n)$ is the greatest k such that $2^k \mid n$, and $\mu(n) = \lfloor \log_2(n) \rfloor$ (and of course, the main intuition is that, if we are going to think of n as a finite set, then $\lambda(n)$ and $\mu(n)$ are simply the minimum and maximum elements, respectively). The main properties of the functions λ, μ that we will use here (all of which can be proven easily and elementarily) are the following.

Proposition 4.3.

(1) The function μ is finite-to-one (in fact, for a given k there are exactly 2^k distinct values of n such that $\mu(n) = k$),
(2) if $\mu(n) < \mu(m)$ then $\mu(n+m) = \mu(m)$,

(3) *if* $\mu(n) = \mu(m)$ *then* $\mu(m+n) = \mu(n) + 1$,
(4) *if* $\lambda(n) < \lambda(m)$ *then* $\lambda(n+m) = \lambda(n)$,
(5) *if* $\lambda(n) = \lambda(m) = k$ *then* $\lambda(n+m) > k$. □

Lemma 4.4. *Let u be a $\mathbb{Z}$-Ramsey ultrafilter. Then, there is a set $X \in u$ such that, for all $a, b, c \in X$ with $a < b < c$,*

(1) $\mu(c-b) \neq \mu(b-a)$, *and*
(2) $\lambda(c-b) \neq \lambda(b-a)$.

Proof. Consider the $\mathbb{Z}$-invariant colouring $c : [\mathbb{N}]^2 \longrightarrow 2^3$ given by defining $c(x,y) = (i,j,k)$ where $i \equiv \mu(y-x) \mod 2$, $j \equiv \lambda(y-x) \mod 2$, and k is the $(\lambda(y-x)+1)$-st binary digit of $y-x$ (equivalently, if $y - x = 2^{\lambda(y-x)}z$, where z is odd, then $2k+1 \equiv z \mod 4$). By hypothesis, u is a $\mathbb{Z}$-Ramsey ultrafilter, so there exists a c-monochromatic set $X \in u$; we claim that X is as sought. To see this, take any $a, b, c \in X$ with $a < b < c$. Then, if we had $\mu(c-b) = \mu(b-a) = m$, we would have $\mu(c-a) = \mu((c-b)+(b-a)) = m+1$, and so the parity of $\mu(c-a)$ is opposite to that of $\mu(c-b)$, contradicting c-monochromaticity. Similarly, if $\lambda(c-b) = \lambda(b-a) = l$, then we could write $b - a = 2^l A$ and $c - b = 2^l B$, with $A \equiv B \mod 4$ and both A, B odd. This means $A + B \equiv 2 \mod 4$, and therefore, from

$$c - a = (c-b) + (b-a) = 2^l(A+B),$$

we can deduce that $\lambda(c-a) = l+1$, again contradicting c-monochromaticity of l. □

Lemma 4.5. *Let u be any ultrafilter, and let $X \in U$ be such that, for all $a, b, c \in X$, $\lambda(c-b) \neq \lambda(b-a)$ and $\mu(c-b) \neq \mu(b-a)$. If $\{x_n \mid n < \omega\}$ is the increasing enumeration of X, then:*

(1) *for each $a \in D(X)$, there is an $n < \omega$ such that $\lambda(a) = \lambda(x_{n+1} - x_n)$,*
(2) *for each $a \in D(X)$, there is an $n < \omega$ such that $\mu(a) = \mu(x_{n+1} - x_n)$, and*
(3) *the function $n \longmapsto \lambda(x_{n+1} - x_n)$ is finite-to-one.*

Proof. We prove points (1) and (2) at once, so let κ denote either μ or λ, accordingly. Given any $a \in D(X)$, there are $n < m$ such that $a = x_m - x_n$. The proof is by induction on $m - n$; there is nothing to prove if $m - n = 1$. Now suppose $a = x_{m+1} - x_n$ and the result holds for $x_m - x_n$. By hypothesis $\kappa(x_{m+1} - x_m) \neq \kappa(x_m - x_n)$, so $\kappa(x_{m+1} - x_n) = \kappa(x_{m+1} - x_m + x_m - x_n)$ equals either $\kappa(x_{m+1} - x_m)$, or $\kappa(x_m - x_n)$ (the maximum of those two if $\kappa = \mu$; the minimum if $\kappa = \lambda$). In the former case, we are done; in the latter, by induction hypothesis there is $k < \omega$ such that $\kappa(x_{m+1} - x_n) = \kappa(x_m - x_n) = \kappa(x_{k+1} - x_k)$, and we are done.

Now to prove point (3), we will prove by induction on k that there are only finitely many $n < \omega$ such that $\lambda(x_{n+1} - x_n) = k$. So suppose the result is true for all $k' < k$; then, there exists an n' such that whenever $n > n'$, we have $\lambda(x_{n+1} - x_n) \geq k$. Assume there are two distinct n, m with $n' < n < m$ such that $\lambda(x_{m+1} - x_m) = \lambda(x_{n+1} - x_n) = k$; suppose further that n, m were chosen so that $m - n$ is minimal. Then, $\lambda(x_{t+1} - x_t) > k$ for all $n < t < m$. We thus have

$$\begin{aligned} \lambda(x_m - x_n) &= \lambda(x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n) \\ &= \lambda(x_{n+1} - x_n) = k, \end{aligned}$$

so that $k = \lambda(x_{m+1} - x_m) = \lambda(x_m - x_n)$, a contradiction. Therefore, there is at most one value of $n > n'$ with $\lambda(x_{n+1} - x_n) = k$; overall, there are only finitely many t with $k = \lambda(x_{t+1} - x_t)$. □

Theorem 4.6. *If u is a $\mathbb{Z}$-Ramsey ultrafilter, then $\mu(D(u))$ is a P-point.*

Proof. Let $f:\omega\longrightarrow\omega$ be an arbitrary function. We will show that f is either constant, or finite-to-one, on a set in $\mu(D(u))$. Consider the partition $\omega=A_0\cup A_1$, where

$$\begin{aligned} A_0 &= \{n\in\mathbb{N}\mid f(\mu(n))<\lambda(n)\},\\ A_1 &= \{n\in\mathbb{N}\mid \lambda(n)\le f(\mu(n))\},\end{aligned}$$

let $i\in 2$ be such that $A_i\in D(u)$, and let $X\in u$ be such that $D(X)\subseteq A_i$; by Lemma 4.4, we may assume that whenever $a,b,c\in X$ with $a<b<c$, we have $\mu(c-b)\neq\mu(b-a)$ and $\lambda(c-b)\neq\mu(b-a)$. Let $X=\{x_n\mid n<\omega\}$ be the increasing enumeration of X, and note that there are two cases

Case $A_1\in u$**:** In this case, we will show that $f\restriction\mu[D(X)]$ is finite-to-one. By Lemma 4.5 (2) we have $\mu[D(X)]=\{\mu(x_{n+1}-x_n)\mid n<\omega\}$. Now, given a $k<\omega$, by Lemma 4.5 (3) there is an N such that for $n\ge N$ we have $\lambda(x_{n+1}-x_n)>k$; for each such n we may thus conclude that $f(\mu(x_{n+1}-x_n))\ge\lambda(x_{n+1}-x_n)>k$ since $D(X)\subseteq A_1$. That is, $f^{-1}[\{k\}]\cap D(X)$ is finite (of cardinality at most N); since the previous reasoning holds for all k, we conclude f is finite-to-one in $\mu[D(X)]$.

Case $A_0\in u$**:** In this case we will find a $Z\in u$ such that f is constant in $\mu[D(Z)]$. By Lemma 4.5 (3) and possibly dropping finitely many elements of X, we may assume without loss of generality that $k_0=\lambda(x_1-x_0)<\lambda(x_{n+1}-x_n)$ for all $n<\omega$. From here, one may easily prove by induction on $n<\omega$ that $\lambda(x_n-x_0)=k_0$ for all $n\ge 2$. Now let $Y=\{x_n\mid n\ge 2\}$, and consider the partition $D(Y)=B_0\cup B_1$ given by

$$\begin{aligned} B_0=\big\{a\in D(Y) \quad &| \quad \text{for some } m>n\ge 2,\ a=x_m-x_n\\ &\qquad \text{and } \mu(x_m-x_n)>\mu(x_n-x_0)\big\},\\ B_1=\big\{a\in D(Y) \quad &| \quad \text{for all } m>n\ge 2 \text{ such that } a=x_m-x_n,\\ &\qquad \mu(x_m-x_n)<\mu(x_n-x_0)\big\}.\end{aligned}$$

There is $j\in 2$ such that $B_j\in D(u)$; we will now argue that $j=1$ is impossible. Suppose $B_1\in D(u)$, and let $W\subseteq Y$ be such that $D(W)\subseteq B_1$. Pick an increasing sequence $n_1<n_2<\cdots$ such that each $x_{n_t}\in W$ and $x_{n_{t+1}}-x_{n_t}<x_{n_{t+1}}-x_{n_{t+1}}$; this way we ensure that thet set $B=\{x_{n_{t+1}}-x_{n_t}\mid t<\omega\}$ is infinite, and a subset of $D(W)$. Letting $k=\mu(x_{n_0}-x_0)$, we now claim that, for all t, we have $\mu(x_{n_{t+1}}-x_{n_t})<k$. To see this, note that $\mu(x_{n_{t+1}}-x_{n_t})<\mu(x_{n_t}-x_0)$, by the definition of B_1; from there one can prove inductively that $\mu(x_{n_t}-x_0)=k$ for all t and conclude the claim. But this is a contradiction since B is infinite but μ is a finite-to-one function. Hence we conclude that $B_0\in D(u)$. Now, if $a\in B_0$, then there are $n<m$ such that $a=x_m-x_n$ and $\mu(x_m-x_n)>\mu(x_n-x_0)$. This implies that $\mu(a)=\mu(x_m-x_n)=\mu(x_m-x_n+x_n-x_0)=\mu(x_m-x_0)$. Thus, by the definition of A_0 we must have $f(\mu(a))=f(\mu(x_m-x_0))<\lambda(x_m-x_0)=k_0$, hence f is bounded (by k_0) in $\mu[B_0]$ and so, partitioning $\mu[B_0]=\bigcup_{s<k_0}(\mu[B_0]\cap f^{-1}[\{s\}])$, we conclude that $f^{-1}[\{s\}]\in\mu(D(u))$ for some $s<k_0$, that is, the function f is constant on some element of $\mu(D(u))$.

The treatment of the two (exhaustive) cases finishes the proof that $\mu(D(u))$ is a P-point. □

Corollary 4.7. *It is relatively consistent with* ZFC *that there are no $\mathbb{Z}$-Ramsey ultrafilters.* □

5. Some $\mathbb{Z}$-Ramsey ultrafilters are Ramsey

In this section we prove two results stating certain conditions under which some $\mathbb{Z}$-Ramsey ultrafilters must necessarily be selective. Whether this implication holds in general remains an open problem.

We first improve on Petrenko and Protasov's result [11] that every $(\mathbb{Z}, 4)$-Ramsey ultrafilter is selective.

Theorem 5.1. *If an ultrafilter $u \in \beta\mathbb{Z}$ is $(\mathbb{Z}, 3)$-Ramsey, then it is selective.*

Proof. Define the colouring $c : [\mathbb{Z}]^3 \longrightarrow 2$ given by $c(\{x, y, z\}) = 1$ iff $z - y > y - x$, whenever $x < y < z$. Let $A \in u$ be a set such that $[A]^3$ is c-monochromatic. Note that, by picking arbitrary $x, y \in A$ with $x < y$, (since A is unbounded) we can always find a $z \in A$ such that $z - y > y - x$, which means that $c(\{x, y, z\}) = 1$ and so $[A]^3$ is monochromatic in colour 1.

Suppose we enumerate increasingly the elements of A as $\{x_n \mid n < \omega\}$. The fact that $[A]^3$ is monochromatic in colour 1 means that $x_2 - x_1 > x_1 - x_0$ and, in general, $x_{n+1} - x_n > x_n - x_1$. That is, the distance between each element of A and the next one is strictly greater than any other distance between two lesser elements. This is easily seen to imply that A must be a thin set, and so the result follows by Theorem 3.2 □

Having proved Theorem 5.1, it is no longer necessary to consider the notion of $(\mathbb{Z}, n)$-Ramsey ultrafilters for $n \geq 3$ (as this is already equivalent to the notion of a selective ultrafilter). Therefore, to simplify notation, we will from now on simply write $\mathbb{Z}$-Ramsey instead of $(\mathbb{Z}, 2)$-Ramsey.

The next theorem to prove, which is the main result of this section, is that every $\mathbb{Z}$-Ramsey ultrafilter that is also a P-point must be a selective ultrafilter. The remainder of the section is devoted to prove this result.

Lemma 5.2. *Let u be a $\mathbb{Z}$-Ramsey ultrafilter. Then, there exists an $A \in u$ without arithmetic progressions of length three.*

Proof. By Lemma 4.4, there is an $A \in u$ such that, whenever $a, b, c \in A$ with $a < b < c$, we have $\mu(c - b) \neq \mu(b - a)$. In particular, $c - b \neq b - a$, so the elements a, b, c do not form an arithmetic progression. □

Recall that, for subsets $A, B \subseteq \mathbb{N}$, we denote $D(A, B) = \{y - x \mid x \in A, y \in B,$ and $x < y\}$. Similarly, for an $x_0 \in \mathbb{N}$ we use the notation $D(x_0, A) = \{y - x_0 \mid y \in A$ and $x_0 < y\}$. Note, e.g., that if we have a set A containing an $x_0 \in A$ with $D(x_0, A) \cap D(B, B) \neq \varnothing$, where $B = A \setminus (x_0 + 1)$, then A is not a thin set.

Lemma 5.3. *Let u be a $\mathbb{Z}$-Ramsey ultrafilter, let $x_0 \in \mathbb{N}$ and let $A \in u$. Then, there exists a set $B \in u$, with $B \subseteq A$, such that for every two distinct $y, z \in B$, it must be the case that $x_0 + z - y \notin B$ (in other words, $D(B, B) \cap D(x_0, B) = \varnothing$).*

Proof. First, use Lemma 5.2 to obtain an $A' \subseteq A$, $A' \in u$ containing no nontrivial arithmetic progressions. Define now a colouring $c : [A']^2 \longrightarrow 2$ by the formula

$$c(\{x,y\}) = \begin{cases} 1; & \text{if } x_0 + y - x \in A' \text{ (that is, if } y - x \in D(x_0, A')) \\ 0; & \text{otherwise.} \end{cases}$$

Note that c is $\mathbb{Z}$-invariant and, therefore, there exists a $B \subseteq A'$, with $B \in u$, such that $[B]^2$ is c-monochromatic. We claim that B is as required in the statement of the lemma. This is clearly the case, by the definition of c, in case $[B]^2$ is monochromatic in colour 0 (since $D(x_0, B) \subseteq D(x_0, A')$). So we assume without loss of generality that $[B]^2$ is monochromatic in colour 1; since u is nonprincipal, assume also that $x_0 < \min(B)$. Aiming for a contradiction, suppose that $D(B,B) \cap D(x_0, B) \neq \varnothing$, and pick $x, y, z \in B$ such that $z - y = x - x_0$. Since x_0 is the smallest of the four numbers under consideration, necessarily z must be the largest. Note also that we may assume without loss of generality that $x < y < z$ (otherwise, if $x_0 < y < x < z$, then we also have that $z - x = y - x_0$ and so we might simply relabel x and y). In particular, $x, y, z \in A'$; since $x, y \in B$, then $c(\{x,y\}) = 1$ and so we must have $w = x_0 + y - x \in A'$. But then $y - w = x - x_0 = z - y$, which implies that $\{w, y, z\}$ is a nontrivial arithmetic progression in A', a contradiction. □

Corollary 5.4. *Let u be a $\mathbb{Z}$-Ramsey ultrafilter, let $F \in [\mathbb{N}]^{<\omega}$, and let $A \in u$. Then, there is a $B \in u$, $B \subseteq A$, such that there are no distinct $y, z \in B$ and $w \in B$, $x \in F$ such that $z - y = w - x$ (in other words, such that $D(B,B) \cap D(F,B) = \varnothing$).*

Proof. Induction on $|F|$, the case $|F| = 1$ being Lemma 5.3 and, for $|F| > 1$, it suffices to pick $x_0 \in F$, use A' as in the conclusion of the Corollary for $F \setminus \{x_0\}$, and use again Lemma 5.3 to get a further $B \subseteq A'$, $B \in u$, satisfying the desired property. □

Lemma 5.5. *Let u be a $\mathbb{Z}$-Ramsey ultrafilter, and let $d \in \mathbb{N}$ be arbitrary. Then, there exists an $A \in u$ such that any distance between two elements of A is greater than d (i.e., for any distinct $x, y \in A$ with $x < y$, we have $y - x > d$).*

Proof. Consider the $\mathbb{Z}$-invariant colouring $c : [\mathbb{Z}]^2 \longrightarrow 2$ given by $c(\{x,y\}) = 1$ iff $|y - x| > d$. Any $A \in u$ such that $[A]^2$ is c-monochromatic will be monochromatic in colour 1 (since A is infinite and hence unbounded in $\mathbb{N}$). Therefore the set A is as required. □

Recall the P-point game for a nonprincipal ultrafilter u. Players I and II alternate plays; in the n-th inning, player I chooses an $A_n \in u$ and then player II chooses a finite $F_n \subseteq A_n$; in the end, player II wins the game if and only if $\bigcup_{n<\omega} F_n \in u$. It is a classical folklore result (attributed to Galvin and McKenzie by Shelah in Chapter VI of [13]) that a nonprincipal ultrafilter u is a P-point if and only if the P-point game for u is not determined (equivalently, u is a P-point if and only if I does not have a winning strategy for this game, since it is easily shown that player II never has a winning strategy).

Theorem 5.6. *Let u be a $\mathbb{Z}$-Ramsey ultrafilter that is also a P-point. Then, u is a selective ultrafilter.*

Proof. We describe a strategy for player I in the P-point game for u. Begin by playing $A_0 = \mathbb{N}$ and, in the $n+1$-th inning, knowing player II's previous moves $F_0, \ldots, F_n$, satisfying $\max(F_i) < \min(F_{i+1})$ for $i < n$, we play an $A_n \in u$ such that

$\min(A_n) > \max(F_n)$, $x, y \in A_n$ and $x < y$ implies $y - x > 2\max(F_n)$ (which can be done by Lemma 5.5) and also such that $D(A_{n+1}, A_{n+1}) \cap D(F_n, A_{n+1}) = \varnothing$ (which can be done by Corollary 5.4). Since u is a P-point, the strategy we just described is not a winning strategy, so there is a run of the game, $\langle F_n \mid n < \omega\rangle$, such that $A = \bigcup_{n<\omega} F_n \in u$.

Define $c : [A]^2 \longrightarrow 2$ given by:

$$c(\{x, y\}) = \begin{cases} 0, & \text{if } x, y \in F_n \text{ for some } n < \omega; \\ 1, & \text{if } x \in F_n \text{ and } y \in F_m \text{ for } m \neq n. \end{cases}$$

Note that c is a $\mathbb{Z}$-invariant colouring: for if $x, y, z, w \in A$ are such that $y - x = w - z$ and, say, x is the least of the four numbers, then if $x \in F_n$ we have two cases:

If $y \in F_n$: then it is not possible to have $w, z \in A \setminus F_n \subseteq A_{n+1}$ by construction, so $z \in F_n$ but then it is impossible that $w \in A \setminus F_n \subseteq A_{n+1}$, so it must be the case that $x, y, z, w \in F_n$ and $c(\{x, y\}) = 0 = c(\{z, w\})$.

If $y \in F_m$ for $m \neq n$: (in which case, we must have $m > n$). Then it is impossible that $w, z \in F_n$, or that $z, w \in A \setminus F_n \subseteq A_{n+1}$, so basically the only option is $z \in F_n$, $w \in A \setminus F_n$ and therefore $c(\{x, y\}) = 1 = c(\{z, w\})$.

Note, also, that any set X that is c-monochromatic is either monochromatic in colour 0 and finite (contained within one single F_n), or infinite and thus monochromatic in colour 1, in which case X is a selector for the family $\{F_n \mid n < \omega\}$. Therefore, if we pick an $X \in u$, with $X \subseteq A$ and such that $[X]^2$ is c-monochromatic, then X must be a selector for the family $\{F_n \mid n < \omega\}$. Thus, by construction, if $x, y, z, w \in X$ with $x < y$, $z < w$, and $x < z$, then if $x \in F_n$, we must have that $x, z, w \in A \setminus F_n \subseteq A_{n+1}$, which makes impossible to have $y - x = w - z$. Therefore, X is a thin set, which implies, by Theorem 3.2, that u is a selective ultrafilter. □

6. A partition result for translation-invariant colourings of the integers

In this section we will complement the results from Section 2 and end with a question.

We begin by recalling the version of the Central Sets Theorem that we will use in this section (this particular formulation can be found in [2], for a proof see [3, Prop. 8.21]).

Theorem 6.1. *Let A be a central set. Then, whenever we have finitely many sequences $\langle a_n^1 \mid n < \omega\rangle, \cdots, \langle a_n^k \mid n < \omega\rangle$, there exists a sequence $\langle x_n \mid n < \omega\rangle$ and a block sequence $\langle H_n \mid n < \omega\rangle$ of finite subsets of ω such that, for every $t \in \{1, \ldots, k\}$ we have*

$$\mathrm{FS}\left(x_n + \sum_{i \in H_t} a_t^i \mid n < \omega\right) \subseteq A.$$

In fact, sets satisfying the conclusion of the previous Theorem are called J-sets [4, Def.14.8.1]. A key result relating these combinatorially rich sets is that every piecewise syndetic set (in particular, every central set) is a J-set [4, Theorem 14.8.3]. Here we focus on central sets, defined as sets that belong to a minimal idempotent element of $\beta\omega$, since they will be the ones that we utilize in our proof.

The main theorem of the section is the following.

Theorem 6.2. *For every $\mathbb{Z}$-invariant colouring $c : [\mathbb{Z}]^2 \longrightarrow r$ there exists a c-monochromatic set X such that $X \notin k\text{-thin}$ for all k.*

The remainder of the section will be used to prove this theorem. For this we introduce the following definitions.

Definition 6.3. Let $s = \langle d, a_1, \dots, a_l \rangle$ be a sequence of natural numbers of length $l + 1$.

(1) Given a triple (i, k, t) of elements of ω, we define

$$(i, k, t) * s = a_{i+1} + a_{i+2} + \cdots + a_{i+k} + td,$$

with the convention that, if $k = 0$, then $(i, k, t) * \vec{a} = td$.

(2) The **l-pattern generated by** s is the set

$$L(s) = \{(i, k, t) * \vec{a} \mid 0 \leq k \leq l, 0 \leq i < l, \text{ and } k - 1 \leq t \leq k + 1\}.$$

The motivation for these definitions comes from an analysis of k-thin sets and difference sets. More concretely, suppose we have a set X whose elements, enumerated increasingly, are $x_0 < x_1 < \cdots < x_{2l} < x_{2l+1}$; suppose furthermore that there is a sequence $s = (d, a_1, \dots, a_l)$ such that $x_{2k+1} - x_{2k} = d$ and $x_{2k+2} - x_{2k+1} = a_k$ for all k. The fact that so many distances between consecutive elements of X equal d shows that X is not an l-thin set. Furthermore, it is not hard to check that in this case we will have $D(X) = L(s)$. So l-patterns arise naturally from considerations regarding how to build sets that are not l-thin.

Proposition 6.4. *Let $s = (d, a_1, \dots, a_l, a_{l+1})$ be an $(l + 2)$-sequence, and denote by $t = (d, a_1, \dots, a_l)$ the $(l + 1)$-sequence that results from dropping the last entry of s. Then, the $(l + 1)$-pattern generated by s is given by*

$$L(s) = L(t) \cup \{y + a_{l+1} \mid y \in L(t)\} \cup \{y + a_{l+1} + d \mid y \in L(t)\} \cup \{a_{l+1}, a_{l+1} + d\}.$$

Proof. Straightforward. □

The following theorem ensures that central sets contain l-patterns for arbitrarily large values of l. The proof is done by induction on l, and the inductive hypothesis that we need is a little stronger, so this is reflected in the way we state the theorem.

Theorem 6.5. *Let A be a central set, and let $l < \omega$. Then there exist sequences $\langle d_n \mid n < \omega \rangle, \langle a_n^1 \mid n < \omega \rangle, \dots, \langle a_n^l \mid n < \omega \rangle$ such that, for all $0 \leq i < l, 0 \leq k \leq l, k-1 \leq t \leq k+1$, letting $s_l^n = \langle d_n, a_n^1, \dots, a_n^l \rangle$ and $\vec{x}^{(i,k,t)} = \langle (i, k, t) * s_l^n \mid n < \omega \rangle$, we have $\mathrm{FS}(\vec{x}^{(i,k,t)}) \subseteq A$. In particular (since each element of $L(s^n)$ is the n-th term of one of the sequences $\vec{x}^{(i,k,t)}$), the set A contains l-patterns for arbitrarily large l.*

Proof. The proof goes by induction on l. For $l = 0$ we only need to take a sequence $\langle d_n \mid n < \omega \rangle$ such that $\mathrm{FS}(\langle d_n \mid n < \omega \rangle) \subseteq A$, which is simply Hindman's theorem (since A is central, in particular it is an IP-set).

Now suppose the theorem already holds for some l and let $\langle \delta_n \mid n < \omega \rangle, \langle \alpha_n^1 \mid n < \omega \rangle, \dots, \langle \alpha_n^l \mid n < \omega \rangle$ be the sequences as in the statement of the theorem. We now apply the Central Sets Theorem 6.1 to the set A along with the sequences

$$\langle (i, k, t) * s_l^n + \delta_n \mid n < \omega \rangle$$

and

$$\langle (i, k, t) * s_l^n + 2\delta_n \mid n < \omega \rangle$$

for each (i,k,t) such that $1 \leq i < l$, $0 \leq k \leq l$, and $k-1 \leq t \leq k+1$; this yields a sequence $\vec{x} = \langle x_n \mid n < \omega \rangle$ of numbers and a block sequence $\langle H_n \mid n < \omega \rangle$ of finite sets satisfying the statement of Theorem 6.1. Define, for each $n < \omega$,

$$\begin{aligned} d_n &= \sum_{i \in H_n} \delta_i, \\ a_n^j &= \sum_{i \in H_n} \alpha_i^j, \quad \text{for } 1 \leq j \leq l, \\ a_n^{l+1} &= x_n + \sum_{i \in H_n} \delta_i. \end{aligned}$$

We now show that, with $\vec{x}^{(i,k,t)}$ defined as in the statement of the theorem for all of the $l+2$ recently obtained sequences, we have $\mathrm{FS}(\vec{x}^{(i,k,t)}) \subseteq A$. If $i+k \leq l$ then the desired result follows immediately by the inductive hypothesis. On the other hand, if $i+k = l+1$, then by looking at the terms of $\vec{x}^{(i,k,t)}$ we obtain

$$\vec{x}^{(i,k,t)} = \begin{cases} \langle x_n + \sum_{i \in H_n} \delta_i \rangle, & \text{if } (i,k,t) = (l,1,0), \\ \langle x_n + \sum_{i \in H_n} 2\delta_i \rangle, & \text{if } (i,k,t) = (l,1,1), \\ \langle x_n + \sum_{i \in H_n} ((i,k-1,t) * s_l^n + \delta_i) \rangle, & \text{if } i \leq l \text{ and } t \leq k, \\ \langle x_n + \sum_{i \in H_n} ((i,k-1,t) * s_l^n + 2\delta_i) \rangle, & \text{if } i \leq l \text{ and } t = k+1. \end{cases}$$

All of these sequences have their set of finite sums contained within A, by the choice of the sequence of x_n utilizing the Central Sets Theorem. □

With the previous theorem in hand, we are in good shape for finally providing a proof of Theorem 6.2.

Proof of Theorem 6.2. Given a $\mathbb{Z}$-invariant colouring $c : [\mathbb{Z}]^2 \longrightarrow 2$, define a colouring of the natural numbers $\tilde{c} : \mathbb{N} \longrightarrow 2$ by $\tilde{c}(n) = c(0,n)$ (so that also $\tilde{c}(n) = c(k, n+k)$ for every k due to the $\mathbb{Z}$-invariance of c). Let u be a minimal idempotent in $\beta\mathbb{N}$, and let $A \in u$ be $\tilde{c}$-monochromatic. Then A is a central set. We will find a set $X \subseteq \mathbb{Z}$ such that $D(X)$ contains arbitrarily long l-patterns, and such that $D(X) \subseteq A$; this will imply on the one hand that X is not k-thin for any k, and on the other hand that $c[[X]^2] = \tilde{c}[D(X)] \subseteq A$ (and hence X will be c-monochromatic). To do this, we recursively define an increasing sequence of finite sets X_l, containing some l-pattern, such that $D(X_l) \subseteq A^* = \{n \in A \mid A - n \in u\}$. In the first step of the induction, it suffices to let X_0 be any set with two elements whose distance belongs to A^*. Now assume we already have our finite set X_{l-1} with $D(X_{l-1}) \subseteq A^*$. Then the set

$$B = A^* \cap \left(\bigcap_{y \in D(X_{l-1})} (A - y) \right)$$

belongs to u, and is therefore central. By Theorem 6.5 there is a sequence $s = (d, a_1, \ldots, a_l)$ such that $L(s) \subseteq B$. Arbitrarily pick numbers $\max(X_{l-1}) < x_0 < x_1 < \cdots < x_2 l < x_{2l+1}$ such that $x_{2k+1} - x_{2k} = d$ and $x_{2k+2} - x_{2k+1} = a_k$ for all k, and let $X_l = X_{l-1} \cup \{x_0, \ldots, x_{2l+1}\}$. It is readily checked that each element of$D(X_l)$ is either an element of $L(s)$ (any distance between two of the x_k), or an element of $D(X_{l-1})$ (any distance between two elements of X_{l-1}), or an element of the form $y + x$ with $y \in D(X_{l-1})$ and $x \in L(s)$ (any distance between an element of X_{l-1} and one of the x_k). In either case (the first case by the choice of B and s, the second case by induction hypothesis, and the third case because $L(s) \subseteq A - y$ for any $y \in D(X_{l-1})$ by construction) we can conclude that $D(X_l) \subseteq A^*$, and the induction can continue. In the end, it suffices to make $X = \bigcup_{n<\omega} X_l$. □

Question 6.6. *Is it the case that for every $\mathbb{Z}$-invariant colouring $c : [\mathbb{Z}]^2 \longrightarrow 2$ there exists a c-monochromatic set $X \in \mathcal{I}^+_{k\text{-thin}}$, that is,*

$$\mathbb{Z} \xrightarrow{\text{left-inv.}} (\mathcal{I}^+_{k\text{-thin}})^2_2\,?$$

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